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MATHEMATICAL PROGRAMMING OVER THE SOLUTION SET OF THE MINIMIZATION PROBLEM FOR THE SUM OF TWO CONVEX FUNCTIONS

CHIH-SHENG CHUANG, LAI-JIU LIN, AND ZENN-TSUN YU

Dedicated to Professor Tomás Domínguez Benavides on the occasion of his 65th birthday

Abstract. In this paper, we study the mathematical programming over the solution set of the minimization problem for the sum of two convex functions, then we apply the result of this problem to study various types of constraint optimization problems, multiple sets split feasibility problems, convex feasibility recover problems, inconsistent feasibility problems, quadratic signal recovery problems, and lasso problems. Our results will have many applications in optimization, signal recovery problem, and image debluring problem. Our results and approach are different from any existence results in the literature.

1. Introduction

Let H_1 and H_2 be real Hilbert spaces with inner product $\langle\cdot,\cdot\rangle$ and norm $||\cdot||$, and let C be nonempty closed convex subset of H_1. The well-known variational inequality problem for an operator $F : C \to H_1$ is the following problem:

$$\text{Find } \bar{x} \in C \text{ such that } \langle F\bar{x}, y - \bar{x}\rangle \geq 0 \text{ for all } y \in C.$$

We denote the solution set of the variational inequality problem by $VI(C, F)$.

Let C and Q be nonempty closed convex subsets of H_1 and H_2, respectively, and let I_1 and I_2 denote the identity mappings in H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded and linear operator with adjoint A^*. Let $\psi : H_1 \to \mathbb{R}$ be convex and Fréchet differentiable on H_1 with Fréchet derivative $\psi^{'}$. Suppose $\psi^{'}$ is κ-Lipschitiz and η-strongly monotone over $T(H_1)$. Let $g_1 : H_1 \to (-\infty, \infty)$ be a convex Fréchet differentiable function with Fréchet derivative ∇g_1 on H_1, and $h_1 : H_1 \to (-\infty, \infty]$ be a proper, convex, and lower semicontinuous function. For each $i = 1, 2, \dots, m$, let $C_1, C_2, \dots, C_m$ be closed convex subsets of H_1. For each $j = 1, 2, \dots, \ell$, let $S_1, S_2, \dots, S_\ell$ be closed convex subsets of H_2. Throughout this paper, we use these notations and assumptions unless specified otherwise.

In this paper , we consider the following problem which contains many optimization problems, nonlinear problems as special cases and it has many applications in optimization theory, image recovery and signal process problems:

Problem (A) Mathematical programming over the solution set of the minimization problem for the sum of two convex functions:

$$\text{Find } \arg\min_{y\in D} \psi(y), \text{ where } D = \arg\min_{x\in H_1}(h_1 + g_1)(x).$$

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Problem (A) contains the following problems as special cases.

(A-1) If we set $\psi(x) = \frac{1}{2}\|x\|^2$, then Problem (A) reduces to the problem:

$$\text{Find } \arg\min_{y \in \arg\min_{x\in H_1}(h_1+g_1)(x)} \|y\|.$$

Problem (A-1) has been studied by Chuang, Yu, and Lin [7], Combettee and Wajs [11], Wang and Xu [16], and Xu [20].

(A-2) If we set $g_1 = 0$, then problem (A) reduces to the bilevel problem with optimization of a convex function as a lower level problem:

$$\text{Find } \arg\min_{y \in \arg\min_{x\in H_1} h_1(x)} \psi(y).$$

(A-3) If we set $h_1 = \iota_C$ (the indicator function of C), then problem (A) reduces to the bilevel problem with optimization of a convex differential function as a lower level problem:

$$\text{Find } \arg\min_{y \in \arg\min_{x\in C} g_1(x)} \psi(y).$$

(A-4) If we set $\psi(x) = \frac{1}{2}\|Ax - b\|^2$, then Problem (A) reduces to least square problem with mathematical program of the sum of two convex functions as a lower level constraint:

$$\text{Find } \arg\min_{y\in D} \tfrac{1}{2}\|Ay - b\|^2, \text{ where } D = \arg\min_{x\in H_1}(h_1 + g_1)(x).$$

(A-5) Mathematical programming with solution of the inconsistent feasibility constraint [8]:
Find $\arg\min_{x\in D} \psi(x)$, where $\alpha_i \geq 0$, $\beta_j \geq 0$ for each $i = 1, 2, \dots, m$, and $j = 1, 2, \dots, \ell$, and $\sum_{i=1}^m \alpha_i + \sum_{j=1}^{\ell} \beta_j = 1$, and

$$D = \arg\min_{x\in H_1} \sum_{i=1}^m \alpha_i \frac{1}{2} d(x, C_i)^2 + \sum_{j=1}^{\ell} \beta_j \frac{1}{2} d(Ax, S_j)^2.$$

The constraint set of an optimization is inconsistent due to wrong assumption, wrong information on the constraint or has some difficulty to describe the constraint. This type of problem often occurs in real world. Therefore it is important to study this problem.

(A-6) If we set $\psi(x) = \frac{1}{2}\|x\|^2$, then Problem (A-5) reduces to minimum norm solution of the inconsistent feasibility problem [8]:

$$\text{Find } \arg\min_{y \in \arg\min_{x\in H_1} \sum_{i=1}^m \alpha_i \frac{1}{2} d(x,C_i)^2 + \sum_{j=1}^{\ell} \beta_j \frac{1}{2} d(Ax,S_j)^2} \|y\|,$$

where $\alpha_i \geq 0$, $\beta_j \geq 0$ for each $i = 1, 2, \dots, m$, and $j = 1, 2, \dots, \ell$, and $\sum_{i=1}^m \alpha_i + \sum_{j=1}^{\ell} \beta_j = 1$.

(A-7) If $\bigcap_{i=1}^m C_i \neq \emptyset$, then Problem (A-6) reduces to the convexly constrained pseudoinverse problem [17]:
Find $\arg\min_{x\in D} \psi(x)$, where $D = \bigcap_{i=1}^m \arg\min_{x\in C} \|A_i x - b_i\|$, and $A_i : H_1 \to H_2$ is bounded, $b_i \in H_2$, $i = 1, 2, \dots, m$.

(A-8) If $\{x : x \in \bigcap_{i=1}^m C_i, Ax \in \bigcap_{j=1}^{\ell} S_j\} \neq \emptyset$, then Problem (A-5) reduces to the optimization problem with multiple set split feasibility constraints:

$$\arg\min_{x\in D} \psi(x), \text{ where } D = \{x : x \in \textstyle\bigcap_{i=1}^m C_i,\ Ax \in \bigcap_{j=1}^{\ell} S_j\}.$$

(A-9) If we set $\psi(x) = \frac{1}{2}\|x\|^2$, then Problem (A-8) reduces to minimum norm solution of the multiple split feasibility problem [5, 18, 21, 22]:

$$\text{Find } \arg\min_{x\in\bigcap_{i=1}^m C_i, Ax\in\bigcap_{j=1}^\ell S_j} \|x\|.$$

The multiple set split feasibility problem contains the following split feasibility problem:

$$\text{Find } \ \bar{x} \in H_1 \text{ such that } \bar{x} \in C \text{ and } A\bar{x} \in Q.$$

The split feasibility problem has many applications in signal processing, image reconstruction, intensity-modulated radiation therapy, approximation theory, control theory, biomedical engineering, communications, and geophysics [2, 3, 4, 18, 19].

(A-10) If $S_j = H_2$ for all $j = 1, 2, \dots, \ell$, then Problem (A-8) reduces to mathematical programming with convex feasibility constraint:

$$\text{Find } \arg\min_{x\in D} \psi(x), \text{ where } D = \{x \in H_1 : x \in \textstyle\bigcap_{i=1}^m C_i\}.$$

(A-11) The general quadratic signal recovery problem [9]:

$$\text{Find } \arg\min_{y\in D_1} \sum_{i=1}^{s} \beta_i \|A_i y - b_i\|^2, \text{ where } D = \{x : x \in \textstyle\bigcap_{i=1}^m C_i\}.$$

(A-12) Optimization over the common solution set of linear equation problems:

$$\arg\min_{x\in D} \psi(x), \text{ where } D_1 = \textstyle\bigcap_{i=1}^m \{x \in H_1 : A_i x = b_i\}.$$

(A-13) Minimum norm solution of the convex feasibility problem:

$$\text{Find } \arg\min_{x\in\bigcap_{i=1}^m C_i} \|x\|.$$

If we let $C_i = \{x \in H_1 : g_i(x) \le t_i$, where $g_i : H_1 \to \mathbb{R}\}$ is a function and $t_i > 0$, then the convex feasibility problem reduces to the signal process and image recovery problem [8].

(A-14) The least absolute shrinkage and select operator problem (in short, Lasso) given by Tibshinani [15] is the problem:

$$\text{Find } \min_x \tfrac{1}{2}\|Ax - b\|_2^2 \text{ subject to } \|x\|_1 \le t,$$

where A be a $m \times n$ real matrix, $x \in \mathbb{R}^n$, $b \in R^m$, $\gamma \ge 0$ be a regularization parameter and $t \ge 0$, $\|x\|_1 = \sum_{i=1}^n |x_i|$ for $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, and $\|y\|_2^2 = \sum_{i=1}^m y_i^2$ for $y = (y_1, y_2, \dots, y_m) \in \mathbb{R}^m$.

In this paper, we study the mathematical programming over the solution set of the minimization problem for the sum of two convex functions, then we apply the result of this problem to study various types of constraint optimization problems, multiple sets split feasibility problems, convex feasibility recover problems, inconsistent feasibility problems, quadratic signal recovery problems, and lasso problems. Our results will have many applications in optimization, signal recovery problem, and image deblurring problem. Our results and approach are different from any existence results in the literature.

2. Preliminaries

Throughout this paper, let $\mathbb{N}$ be the set of positive integers and let $\mathbb{R}$ be the set of real numbers. Let $\Gamma_0(H_1)$ be the space of all proper, lower semi-continuous, and convex functions from H_1 into $(-\infty, \infty]$. We denote the strongly convergence of $\{x_n\}_{n\in\mathbb{N}}$ to $x \in H_1$ by $x_n \to x$.

Let C be a nonempty closed convex subset of a real Hilbert space H_1. Then a mapping $T : C \to H_1$ is called

(i) nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in H_1$;
(ii) firmly nonexpansive if $||Tx - Ty||^2 \leq ||x - y||^2 - ||(I - T)x - (I - T)y||^2$ for all $x, y \in C$;
(iii) Lipschitz continuous if there exists $L > 0$ such that $\|Tx - Ty\| \leq L\|x - y\|$ for all $x, y \in C$;
(vi) monotone if $\langle x - y, Tx - Ty\rangle \geq 0$ for all $x, y \in C$;
(iv) strongly monotone if there exists $\bar{\gamma} > 0$ such that $\langle x-y, Tx-Ty\rangle \geq \bar{\gamma}\|x-y\|^2$ for all $x, y \in C$;
(v) α-inverse-strongly monotone (in short, $\alpha-ism$) if $\langle x-y, Tx-Ty\rangle \geq \alpha\|Tx-Ty\|^2$ for all $x, y \in C$ and $\alpha > 0$.

We also know that if $V : C \to H_1$ is a monotone mapping, then $T = I - V$ is a pseudo-contractive mapping.

Let C be a nonempty closed convex subset of a real Hilbert space H. For each $x \in H$, there is a unique element $\bar{x} \in C$ such that $||x - \bar{x}|| = \min_{y\in C} ||x - y||$. In this study, we set $P_C x = \bar{x}$, and P_C is called the metric projection from H onto C.

Let $B : H_1 \multimap H_1$ be a multivalued mapping. The effective domain of B is denoted by $D(B)$, that is, $D(B) = \{x \in H_1 : Bx \neq \emptyset\}$. Then $B : H_1 \multimap H_1$ is monotone on H_1 if $\langle x - y, u - v\rangle \geq 0$ for all $x, y \in D(B), u \in Bx$, and $v \in By$. B is maximal monotone on H_1 if B is a monotone operator on H_1 and its graph is not properly contained in the graph of any other monotone operator on H_1.

For a maximal monotone operator $B : H_1 \multimap H_1$ and $r > 0$, we may define a single-valued mapping $J_r^B : H_1 \to D(B)$ by $J_r^B = (I + rB)^{-1}$, and it is called the resolvent mapping of B for r. If $B : H_1 \multimap H_1$ is maximal monotone and $\beta > 0$, then J_β^B is single-valued and firmly nonexpansive [14].

A mapping $S : C \to H_1$ is said to be averaged if there exist $\alpha \in (0, 1)$ and a nonexpansive mapping $T : C \to H_1$ such that $S = (1 - \alpha)I + \alpha T$. In this case, we set $T_\alpha = S$ and say that T_α is α-averaged. In fact, a firmly nonexpansive mapping is $\frac{1}{2}$-averaged.

Lemma 2.1 ([10])**.** *Let $S, T : C \to H_1$ be a mapping. Thus,*

(i) *T is nonexpansive if and only if the complement $(I - T)$ is $1/2$-ism.*
(ii) *If S is υ-ism, then for $\gamma > 0$, γS is υ/γ-ism.*
(iii) *S is averaged if and only if the complement $I-S$ is υ-ism for some $\upsilon > 1/2$.*
(iv) *If S and T are both averaged, then the product (composite) ST is averaged.*
(v) *If the mappings $\{T_i\}_{i=1}^n$ are averaged and have a common fixed point, then $\bigcap_{i=1}^n Fix(T_i) = Fix(T_1 \ldots T_n)$.*

Let $f : H_1 \to (-\infty, \infty]$ be a proper, lower-semicontinuous, and convex function. Then the subdifferential ∂f of f is defined by

$$\partial f(x) = \{u \in H_1 : f(y) \geq f(x) + \langle y - x, u\rangle \text{ for all } y \in H_1\}.$$

Let $x \in H_1$, $\mu(x)$ denote the family of all neighborhood of x, let H_1 be a Hilbert space, let $C \in \mu(x)$, and $f : C \to (-\infty, \infty]$ be a function. Then f is said to be Fréchet differentiable at x if there exists an operator $\nabla f(x) \in B(H_1, \mathbb{R})$, call the

Fréchet derivative of f at x, such that

$$\lim_{0\neq\|y\|\to 0} \frac{|f(x+y)-f(x)-\langle y, \nabla f(x)\rangle|}{\|y\|} = 0.$$

Let C be a nonempty closed convex subset of H_1. The indicator function ι_C defined by

$$\iota_C x = \begin{cases} 0, \ x \in C \\ \infty, \ x \notin C \end{cases}$$

is a proper, lower semicontinuous, and convex function. Further, we see $J_\lambda^{\partial i_C} = P_C$.

Let $g \in \Gamma_0(H)$ and $\lambda \in (0,\infty)$. The proximal operator of $g \in \Gamma_0(H)$ of order $\lambda \in (0,\infty)$ is defined as the proximal operator of λg, that is,

$$prox_{\lambda g} x = \arg\min_{v\in H}\{g(v) + \frac{1}{2\lambda}\|v-x\|^2\}, \ x \in H.$$

Lemma 2.2 ([16]). *Let $g \in \Gamma_0(H)$ and $\lambda \in (0,\infty)$. Thus,*

(i) $prox_{\lambda g} = (I+\lambda\partial g)^{-1} = J_\lambda^{\partial g}$;
(ii) $prox_{\lambda g}$ *is firmly nonexpansive;*
(iii) *If C is a nonempty closed convex subset of H and $g = i_C$, then $prox_{\lambda g} = P_C$ for all $\lambda \in (0,\infty)$.*

Lemma 2.3 ([20]). *Let $f, g \in \Gamma_0(H_1)$, $x^* \in H_1$, and $\lambda \in (0,\infty)$. Assume that f is finite valued and Fréchet differentiable function on H_1 with Fréchet derivative ∇f. Then x^* is a solution of the problem* $\arg\min_{x\in H_1} f(x) + g(x)$ *if and only if $x^* = prox_{\lambda g}(I - \lambda\nabla f)x^*$.*

Lemma 2.4 ([13]). *Let C be a nonempty closed convex subset of H_1, and $f : H_1 \to \mathbb{R}$ be convex and Fréchet differentiable with the Fréchet derivative A. Then $VI(C,A) = \arg\min_{x\in C} f(x)$.*

3. Mathematical programming over the solution set of the minimization problem for the sum of two convex functions

Throughout this paper, we say condition (E) is satisfied if the following conditions hold:

$$\text{(i)} \lim_{n\to\infty}\lambda_n = 0; \text{ (ii)} \textstyle\sum_{n=0}^\infty \lambda_n = \infty; \text{ (iii)} \displaystyle\lim_{n\to\infty}\frac{\lambda_n - \lambda_{n+1}}{\lambda_{n+1}^2} = 0.$$

In 2001, Yamada [17] established the following important result for the variational inequality problem over the fixed point set of nonexpansive mapping.

Theorem 3.1 ([17]). *Let $\kappa > 0, \eta > 0, T : H_1 \to H_1$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$ and let $F : H_1 \to H_1$ be a κ-Lipschitz continuous and η-strongly monotone mapping over $T(H_1)$. Let $x_1 \in H_1$, let $\mu \in (0, \frac{2\eta}{\kappa^2})$, and let $\{\lambda_n\}_{n\in\mathbb{N}\cup\{0\}} \subset (0,1]$ be any sequence satisfying condition (E). Let $\{x_n\}_{n\in\mathbb{N}} \subset H_1$ be defined by*

$$x_{n+1} = (I - \mu\lambda_n F)Tx_n$$

for all $n \in \mathbb{N}$. Then $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to the unique solution $\bar{x}$ of $VI(Fix(T), F)$.

By Theorem 3.1 and Lemma 2.4, we get the following result, and it plays an important role in this paper.

Theorem 3.2. *Suppose that* $\psi^{'} : H_1 \to H_1$ *is* κ*-Lipschitz continuous and* η*-strongly monotone over* $T(H_1)$ *and* $Fix(T) \neq \emptyset$. *Let* $x_1 \in H_1$, $\mu \in (0, \frac{2\eta}{\kappa^2})$, *and* $\{\lambda_n\}_{n\in\mathbb{N}} \subset (0, 1]$ *be any sequence satisfying condition (E). Let* $\{x_n\}_{n\in\mathbb{N}} \subset H_1$ *be defined by*

$$x_{n+1} = (I - \mu\lambda_n\psi^{'})Tx_n$$

for all $n \in \mathbb{N}$. *Then* $\{x_n\}_{n\in\mathbb{N}}$ *converges strongly to the unique solution* $\bar{x}$ *of* $\arg\min_{y\in Fix(T)} \psi(y)$.

Proof. Since T is a nonexpansive mapping, $Fix(T)$ is a closed convex set of H_1. Apply Theorem 3.1, $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to the unique solution $\bar{x}$ of $VI(Fix(T), \psi^{'})$. Next, by Lemma 2.4, $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to the unique solution $\bar{x}$ of $\arg\min_{y\in Fix(T)} \psi(y)$. □

Remark 3.3. Theorem 3.2 is different from [12, Theorem 3.7]. Indeed, in Theorem 3.2, we assume that ψ is convex and differentiable on H_1, and $\psi^{'} : H_1 \to H_1$ is κ-Lipschitz continuous and η-strongly monotone over $T(H_1)$. But, in [12, Theorem 3.7], $\psi : H_1 \to \mathbb{R}\bigcup\{\infty\}$ is twice differentiable on some open set U with $\Delta \subset U$ and $\psi^{''} : U \to H_1$ is a bounded operator and is uniformly strongly positive and uniformly bounded over $\triangle$.

Theorem 3.4. *Suppose that* ∇g_1 *is* L*-Lipschitz continuous,* $\arg\min_{x\in H_1}(g_1+h_1)(x) \neq \emptyset$, *and* $\psi^{'} : H_1 \to H_1$ *is* κ*-Lipschitz continuous and* η*-strongly monotone over* $prox_{rh_1}(I - r\nabla g_1)(H_1)$. *Let* $x_1 \in H_1$, $r \in (0, \frac{2}{L})$, $\mu \in (0, \frac{2\eta}{\kappa^2})$, *and any sequence* $\{\lambda_n\}_{n\in\mathbb{N}} \subset (0, 1]$ *satisfying condition (E). Let* $\{x_n\}_{n\in\mathbb{N}} \subset H_1$ *be defined by*

$$x_{n+1} = (I - \mu\lambda_n\psi^{'})prox_{rh_1}(I - r\nabla g_1)x_n$$

for all $n \in \mathbb{N}$. *Then* $\{x_n\}_{n\in\mathbb{N}}$ *converges strongly to the unique solution of* $\arg\min_{y\in D} \psi(y)$, *where* $D = \arg\min_{x\in H_1}(h_1 + g_1)(x)$.

Proof. Let $T = prox_{rh_1}(I - r\nabla g_1)$ in Theorem 3.2. Since ∇g_1 is L-Lipschitz continuous, it follows from [1, Corollary 10] that ∇g_1 is $\frac{1}{L}$-inverse strongly monotone. By Lemma 2.1, $r\nabla g_1$ is $\frac{1}{rL}$-ism and $(I - r\nabla g_1)$ is averaged. Since ∂h_1 is maximum monotone, it follows from Lemma 2.2 that $prox_{rh_1} = J_r^{\partial h_1}$ is firmly nonexpansive. Hence, $prox_{rh_1}$ is $\frac{1}{2}$-averaged. Then by Lemma 2.1, T is averaged and nonexpansive. Further, by Lemma 2.3,

$$Fix(T) = \arg\min_{x\in H_1} g_1(x) + h_1(x).$$

Hence, $Fix(T) \neq \emptyset$, and we get the conclusion of Theorem 3.4 from Theorem 3.2. □

Next, an iteration is used to study the following problem:

$$\arg\min_{y\in D} \tfrac{1}{2}\|Ay - b\|^2, \text{ where } D = \arg\min_{x\in H_1}(h_1 + g_1)(x).$$

Theorem 3.5. *Let* $b \in H_1$. *Suppose that* $\arg\min_{x\in H_1}(g_1 + h_1)(x) \neq \emptyset$ *and* ∇g_1 *is* L*-Lipschitz continuous. Let* $A : H_1 \to H_1$ *be a bounded linear self-adjoint operator on* H_1 *such that* $\eta\|v\|^2 \leq \|Av\|^2$ *for all* $v \in H_1$. *Let* $x_1 \in H_1$, $r \in (0, \frac{2}{L})$, $\mu \in (0, \frac{2\eta}{\kappa^2})$, *and any sequence* $\{\lambda_n\}_{n\in\mathbb{N}} \subset (0,1]$ *satisfying condition (E). Let* $\{x_n\}_{n\in\mathbb{N}} \subset H_1$ *be defined by*

$$x_{n+1} = prox_{rh_1}(I - r\nabla g_1)x_n - \mu\lambda_n A^*(A(prox_{rh_1}(I - r\nabla g_1)x_n) - b)$$

for all $n \in \mathbb{N}$. *Then* $\{x_n\}_{n\in\mathbb{N}}$ *converges strongly to the unique solution of* $\arg\min_{y\in D} \frac{1}{2}\|Ay - b\|^2$, *where* $D = \arg\min_{x\in H_1}(h_1 + g_1)(x)$.

Proof. Let $\psi(y) = \frac{1}{2}\|Ay - b\|^2$. Then it is easy to see that $\psi'(y) = A^*(Ay - b)$. Hence, we know that

$$||\psi'(x) - \psi'(y)|| = ||A^*(Ax - b) - A^*(Ay - b)|| \leq ||A||^2 \cdot ||x - y||$$

for all $x, y \in H_1$, and ψ' is $\|A\|^2$-Lipschitz continuous. We also have

$$\begin{aligned}\langle x - y, \psi'(x) - \psi'(y)\rangle &= \langle x - y, A^*(Ax - b) - A^*(Ay - b)\rangle \\ &= \langle Ax - Ay, (Ax - b) - (Ay - b)\rangle \\ &= ||A(x - y)||^2 \\ &\geq \eta||x - y||^2\end{aligned}$$

for all $x, y \in H_1$, and ψ' is η-strongly monotone. Therefore, Theorem 3.5 follows from Theorem 3.4. □

Next, an iteration is used to study the following bilevel problem:

$$\arg\min_{y\in D} \psi(y), \text{ where } D = \arg\min_{x\in C} g_1(x).$$

Theorem 3.6. *Suppose that* ∇g_1 *is L-Lipschitz continuous,* $\psi' : H_1 \to H_1$ *is* κ*-Lipschitz continuous and* η*-strongly monotone over* $P_C(I - r\nabla g_1)(H_1)$, *and* $\arg\min_{x\in C} g_1(x) \neq \emptyset$. *Let* $x_1 \in H_1$, $r \in (0, \frac{2}{L})$, $\mu \in (0, \frac{2\eta}{\kappa^2})$, *and any sequence* $\{\lambda_n\}_{n\in\mathbb{N}} \subset (0,1]$ *satisfying condition (E). Let* $\{x_n\}_{n\in\mathbb{N}} \subset H_1$ *be defined by*

$$x_{n+1} = (I - \mu\lambda_n\psi')P_C(I - r\nabla g_1)x_n$$

for all $n \in \mathbb{N}$. *Then* $\{x_n\}_{n\in\mathbb{N}}$ *converges strongly to the unique solution* $\bar{x}$ *of* $\arg\min_{y\in D} \psi(y)$, *where* $D = \arg\min_{x\in C} g_1(x)$.

Proof. Let $h_1 = i_C$. By Lemma 2.2, $prox_{\lambda h_1} = P_C$ and

$$\arg\min_{x\in H_1} h_1(x) + g_1(x) = \arg\min_{x\in C} g_1(x).$$

Hence, Theorem 3.6 follows from Theorem 3.4. □

Next, an iteration is used to study the following bilevel problem:

$$\arg\min_{y\in C} \psi(y), \text{ where } D = \arg\min_{x\in H_1} h_1(x).$$

Theorem 3.7. *Suppose that* $\arg\min_{x\in H_1} h_1(x) \neq \emptyset$, *and* $\psi' : H_1 \to H_1$ *is* κ*-Lipschitz continuous and* η*-strongly monotone over* $prox_{rh_1}(H_1)$. *Let* $x_1 \in H_1$, *and* $\mu \in (0, \frac{2\eta}{\kappa^2})$, *and any sequence* $\{\lambda_n\}_{n\in\mathbb{N}} \subset (0,1]$ *satisfying condition (E). Let* $\{x_n\}_{n\in\mathbb{N}} \subset H_1$ *be defined by*

$$x_{n+1} = (I - \mu\lambda_n\psi')prox_{rh_1}x_n$$

for all $n \in \mathbb{N}$. *Then* $\{x_n\}_{n\in\mathbb{N}}$ *converges strongly to the unique solution of* $\arg\min_{y\in D} \psi(y)$, *where* $D = \arg\min_{x\in H_1} h_1(x)$.

Proof. We get the conclusion of Theorem 3.7 when we set $g_1 = 0$ in Theorem 3.4. □

Remark 3.8. For Theorem 3.6 and Theorem 3.7, we know that the properties of the mappings g_1 and h_1 are different, and the iteration process between them are different. So, we give two iteration processes for bilevel mathematical programming under different conditions.

4. The multiple split feasibility problems II, signal process and deblurring problem

Lemma 4.1 ([5]). *Let $A : H_1 \to H_2$ be a bounded linear operator, and Q be a nonempty closed convex subset of H_2. Let $g(x) = \frac{1}{2}\|Ax - P_Q Ax\|_2^2$. Then g is a convex function on H_1 and $\nabla g(x) = A^*(I - P_Q)Ax$ for all $x \in H_1$.*

Next, an iteration is used to study the optimization problem with solution of the inconsistent feasibility constraint:

> $\arg\min_{x\in D} \psi(x)$, where $\alpha_i \geq 0$ and $\beta_j \geq 0$ for each $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, \ell$, and $\sum_{i=1}^m \alpha_i + \sum_{j=1}^\ell \beta_j = 1$, and
>
> $$D = \arg\min_{x\in H_1} \sum_{i=1}^m \frac{\alpha_i}{2} d(x, C_i)^2 + \sum_{j=1}^\ell \frac{\beta_j}{2} d(Ax, S_j)^2.$$

Theorem 4.2. *Suppose that $\psi' : H_1 \to H_1$ is κ-Lipschitz continuous and η-strongly monotone over $(I - r(\sum_{i=1}^m \alpha_i(I - P_{C_i}) - \sum_{j=1}^\ell \beta_j A^*(A - P_{S_j}A)))(H_1)$. For each $i = 1, 2, \ldots, m$ and each $j = 1, 2, \ldots, \ell$, let $\alpha_i \geq 0$, $\beta_j \geq 0$, and $\sum_{i=1}^m \alpha_i + \sum_{j=1}^\ell \beta_j = 1$. Suppose that*

$$D = \arg\min_{x\in H_1} \sum_{i=1}^m \frac{\alpha_i}{2} d(x, C_i)^2 + \sum_{j=1}^\ell \frac{\beta_j}{2} d(Ax, S_j)^2 \neq \emptyset.$$

Let $x_1 \in H_1$, $r \in (0, \frac{2}{1+\|A\|^2})$ and $\mu \in (0, \frac{2\eta}{\kappa^2})$, and any sequence $\{\lambda_n\}_{n\in\mathbb{N}} \subset (0, 1]$ satisfying condition (E). Let $\{x_n\}_{n\in\mathbb{N}} \subset H_1$ be defined by

$$x_{n+1} = (I - \mu\lambda_n\psi')\Big(x_n - r\Big(\sum_{i=1}^m \alpha_i(x_n - P_{C_i}x_n) - \sum_{j=1}^\ell \beta_j A^*(Ax_n - P_{S_j}Ax_n)\Big)\Big)$$

for all $n \in \mathbb{N}$. Then $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to the unique solution of $\arg\min_{y\in D} \psi(y)$.

Proof. For each $i = 1, 2, \ldots, m$ and each $j = 1, 2, \ldots, \ell$, let $f_i(x) = \frac{1}{2}d(x, C_i)^2$ and $r_j(x) = \frac{1}{2}d(Ax, S_j)^2$. Then by Lemma 4.1 and [7, Theorem 5.1], we know that f_i and r_j are convex differential functions, $\nabla f_i(x) = x - P_{C_i}x$, and $\nabla r_j(x) = A^*(Ax - P_{S_j}Ax)$. Let

$$g_1(x) = \sum_{i=1}^m \frac{\alpha_i}{2} d(x, C_i)^2 + \sum_{j=1}^\ell \frac{\beta_j}{2} d(Ax, S_j)^2.$$

Then g_1 is a convex and differentiable function, and

$$\nabla g_1(x) = \sum_{i=1}^{m} \alpha_i (x_n - P_{C_i} x_n) + \sum_{j=1}^{\ell} \beta_j A^*(Ax_n - P_{S_j} Ax_n),$$

and then ∇g_1 is $(1 + \|A\|^2)$-Lipschitz continuous. By Theorem 3.6, $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to the unique solution of $\arg\min_{y\in D} \psi(y)$. $\square$

Next, an iteration is used to study the optimization problem with inconsistent feasibility constraints:

> $\arg\min_{x\in D} \psi(x)$, where $\alpha_i \geq 0$ for each $i = 1, 2, \dots, m$, and $\sum_{i=1}^{m} \alpha_i = 1$, and $D = \arg\min_{x\in H_1} \sum_{i=1}^{m} \frac{\alpha_i}{2} d(x, C_i)^2$.

Corollary 4.3. *Suppose that $\psi' : H_1 \to H_1$ is κ-Lipschitz continuous and η-strongly monotone over $(I - r(\sum_{i=1}^{m} \alpha_i (I - P_{C_i})))(H_1)$. For each $i = 1, 2, \dots, m$, let $\alpha_i \geq 0$ and $\sum_{i=1}^{m} \alpha_i = 1$. Suppose that $D = \arg\min_{x\in H_1} \sum_{i=1}^{m} \frac{\alpha_i}{2} d(x, C_i)^2 \neq \emptyset$. Let $x_1 \in H_1$, $r \in (0, 2)$ and $\mu \in (0, \frac{2\eta}{\kappa^2})$, and any sequence $\{\lambda_n\}_{n\in\mathbb{N}} \subset (0, 1]$ satisfying condition (E). Let $\{x_n\}_{n\in\mathbb{N}} \subset H_1$ be defined by*

$$x_{n+1} = (I - \mu\lambda_n \psi')\Big(x_n - r\Big(\sum_{i=1}^{m} \alpha_i (x_n - P_{C_i} x_n)\Big)\Big)$$

for all $n \in \mathbb{N}$. Then $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to the unique solution of $\arg\min_{y\in D} \psi(y)$.

Proof. Let $H_1 = H_2, A = I, \beta_j = 0$ for all $j = 1, 2, \dots, m$, and $S_j = H_1$. Then Corollary 4.3 follows from Theorem 4.2. $\square$

Next, an iteration is used to study the optimization problem with multiple split feasibility constraints:

> $\arg\min_{x\in D} \psi(x)$, where $\alpha_i \geq 0$, $\beta_j \geq 0$ for each $i = 1, 2, \dots, m$ and for each $j = 1, 2, \dots, \ell$, and $\sum_{i=1}^{m} \alpha_i + \sum_{j=1}^{\ell} \beta_j = 1$, and
>
> $$D_1 = \Big\{x : x \in \bigcap_{i=1}^{m} C_i, Ax \in \bigcap_{j=1}^{\ell} S_j\Big\} \neq \emptyset.$$

Theorem 4.4. *Suppose that $\psi' : H_1 \to H_1$ is κ-Lipschitz continuous and η-strongly monotone over $(I - r(\sum_{i=1}^{m} \alpha_i (I - P_{C_i}) - \sum_{j=1}^{\ell} \beta_j A^*(A - P_{S_j} A))H_1$. For each $i = 1, 2, \dots, m$ and each $j = 1, 2, \dots, \ell$, let $\alpha_i \geq 0$, $\beta_j \geq 0$, and $\sum_{i=1}^{m} \alpha_i + \sum_{j=1}^{\ell} \beta_j = 1$. Suppose that*

$$D_1 = \Big\{x : x \in \bigcap_{i=1}^{m} C_i, Ax \in \bigcap_{j=1}^{\ell} S_j\Big\} \neq \emptyset.$$

Let $x_1 \in H_1$, $r \in (0, \frac{2}{1+\|A\|^2})$ and $\mu \in (0, \frac{2\eta}{\kappa^2})$, and any sequence $\{\lambda_n\}_{n\in\mathbb{N}} \subset (0, 1]$ satisfying condition (E). Let $\{x_n\}_{n\in\mathbb{N}} \subset H_1$ be defined by

$$x_{n+1} = (I - \mu\lambda_n \psi')\Big(x_n - r\Big(\sum_{i=1}^{m} \alpha_i (x_n - P_{C_i} x_n) - \sum_{j=1}^{\ell} \beta_j A^*(Ax_n - P_{S_j} Ax_n)\Big)\Big)$$

for all $n \in \mathbb{N}$. *Then* $\{x_n\}_{n\in\mathbb{N}}$ *converges strongly to the unique solution of* $\arg\min_{y\in D_1} \psi(y)$.

Proof. Let

$$D = \arg\min_{x\in H_1} \sum_{i=1}^{m} \frac{\alpha_i}{2} d(x, C_i)^2 + \sum_{j=1}^{\ell} \frac{\beta_j}{2} d(Ax, S_j)^2.$$

By Theorem 4.2,$\{x_n\}_{n\in\mathbb{N}}$ converges strongly to the unique solution $\bar{x} \in \arg\min_{y\in D} \psi(y)$. Then $\bar{x} \in D$ and $\psi(\bar{x}) \le \psi(y)$ for all $y \in D$. Since $D_1 \ne \emptyset$, there exists $z \in \bigcap_{i=1}^{m} C_i, Az \in \bigcap_{j=1}^{\ell} S_j$ and $g_1(z) = 0 = \min_{x\in H_1} g_1(x)$, where

$$g_1(x) = \sum_{i=1}^{m} \frac{\alpha_i}{2} d(x, C_i)^2 + \sum_{j=1}^{\ell} \frac{\beta_j}{2} d(Ax, S_j)^2.$$

Therefore $z \in D$ and $D_1 \subset D$. Conversely if $w \in D$, then $g_1(w) \le g_1(z) = 0$. Hence, $g_1(w) = 0 = \min_{x\in H_1} g_1(x)$, $d(w, C_i) = 0$ for each $i = 1, 2, \dots, m$, and $d(A(w), S_j) = 0$ for each $j = 1, 2, \dots, \ell$. Then, $w \in \bigcap_{i=1}^{m} C_i, A(w) \in \bigcap_{j=1}^{\ell} S_j, w \in D_1$ and $D \subset D_1$. The proof is completed □

Next, an iteration is used to study the optimization problem with convex feasibility constraints:

$$\arg\min_{x\in D} \psi(x), \text{ where } D = \bigcap_{i=1}^{m} C_i \ne \emptyset.$$

Corollary 4.5. *Suppose that* $\psi' : H_1 \to H_1$ *is* κ*-Lipschitz continuous and* η*-strongly monotone over* $(I - r\sum_{i=1}^{m} \alpha_i(I - P_{C_i}))H_1$. *For each* $i = 1, 2, \dots, m$, *let* $\alpha_i \ge 0$ *and* $\sum_{i=1}^{m} \alpha_i = 1$. *Suppose that* $D = \bigcap_{i=1}^{m} C_i \ne \emptyset$. *Let* $x_1 \in H_1$, $r \in (0, 2)$ $\mu \in (0, \frac{2\eta}{\kappa^2})$, *and any sequence* $\{\lambda_n\}_{n\in\mathbb{N}} \subset (0, 1]$ *satisfying condition (E). Let* $\{x_n\}_{n\in\mathbb{N}} \subset H_1$ *be defined by*

$$x_{n+1} = (I - \mu\lambda_n\psi')(x_n - r\sum_{i=1}^{m} \alpha_i(x_n - P_{C_i}x_n))$$

for all $n \in \mathbb{N}$. *Then* $\{x_n\}_{n\in\mathbb{N}}$ *converges strongly to the unique solution of* $\arg\min_{y\in D} \psi(y)$.

Proof. Let $H_1 = H_2$, $A = I$, $S_j = H_1$, and $\beta_j = 0$ for all $j =, 1, 2, \dots, m$. Then Corollary 4.5 follows from Theorem 4.4. □

Next, an iteration is used to study the general quadratic signal recovery problem:

$$\arg\min_{y\in D_1} \sum_{i=1}^{s} \beta_i \|A_i y - b_i\|^2, \text{ where } D_1 = \bigcap_{i=1}^{m} C_i.$$

Theorem 4.6. *For each* $i = 1, 2, \dots, s$, *let* $A_i : H_1 \to H_1$ *be a bounded linear operator on* H_1, $\eta_i > 0, \beta_i \ge 0, \eta\|v\|^2 \le \|A_i v\|^2$ *for all* $v \in H_1$. *For each* $i = 1, 2, \dots, m$, *let* $\alpha_i \ge 0, \sum_{i=1}^{m} \alpha_i = 1$, *and* $D_1 = \{x : x \in \bigcap_{i=1}^{m} C_i\} \ne \emptyset$. *Let* $x_1 \in H_1$, $r \in (0, 2)$ *and* $\mu \in (0, \frac{2\eta}{\kappa^2})$, *and any sequence* $\{\lambda_n\}_{n\in\mathbb{N}} \subset (0, 1]$ *satisfying condition (E). Let* $\{x_n\}_{n\in\mathbb{N}} \subset H_1$ *be defined by*

$$x_{n+1} = \Big(x_n - r\sum_{i=1}^{m} \alpha_i(x_n - P_{C_i}x_n) - \mu\lambda_n \sum_{i=1}^{s} \beta_i A_i^*(A_i(x_n - r\sum_{i=1}^{m} \alpha_i(x_n - P_{C_i}x_n) - b_i\Big)$$

for all $n \in \mathbb{N}$. *Then* $\{x_n\}_{n\in\mathbb{N}}$ *converges strongly to the unique solution* $\bar{x}$ *of problem* $\arg\min_{y\in D_1} \sum_{i=1}^{s} \beta_i \|A_i y - b_i\|^2$.

Proof. Let $\psi(x) = \frac{1}{2}\sum_{i=1}^{s} \beta_i \|A_i x - b_i\|^2$. Then $\psi'(x) = \sum_{i=1}^{s} \beta_i A_i^*(A_i x - b_i)$. For each $x, y \in H_1$,

$$\|\psi'(x) - \psi'(y)\| \leq \sum_{=1}^{s} \beta_i \|A_i^*(A_i x - A_i y)\| \leq \sum_{i=1}^{s} \beta_i \|A_i\|^2 \|x - y\|,$$

and

$$\begin{aligned}\langle x - y, \psi'(x) - \psi'(y)\rangle =& \langle x - y, \sum_{i=1}^{s} \beta_i A_i^*(A_i x - A_i y)\rangle \\ =& \sum_{i=1}^{s} \beta_i \langle A_i(x-y), A_i(x-y)\rangle = \sum_{i=1}^{s} \beta_i \|A_i(x-y)\|^2 \\ \geq& \sum_{i=1}^{s} \beta_i \eta_i \|x - y\|^2.\end{aligned}$$

This shows that ψ' is $\sum_{i=1}^{s} \beta_i \eta_i$-strongly monotone and $\sum_{i=1}^{s} \beta_i \|A_i\|^2$-Lipschitz continuous. Then Theorem 4.6 follows from Corollary 4.5. □

Remark 4.7. The iteration in Theorem 4.6 is different from the iterations in [9].

Next, an iteration is used to study the generalized convexly constrained pseudoinverse problem:

$$\arg\min\nolimits_{x\in D} \psi(x), \text{ where } D = \bigcap_{i=1}^{m} \arg\min_{x\in C} \|A_i x - b_i\|^2.$$

Theorem 4.8. *For each* $i = 1, 2, \ldots, m$, *let* $A_i : H_1 \to H_2$ *be a bounded linear operator on* H_1, $\alpha_i \geq 0$, $b_i \in H_2$, *and* $D = \bigcap_{i=1}^{m} \arg\min_{x\in H_1} \|A_i x - b_i\|^2 \neq \emptyset$, *and* $f_i(x) = \frac{1}{2}\|A_i x - b_i\|^2$. *Suppose that* $\psi' : H_1 \to H_1$ *is* κ*-Lipschitz continuous and* η*-strongly monotone over* $P_C(I - r_1\nabla f_1)(I - r_2\nabla f_2)\ldots(I - r_\ell\nabla f_\ell)(H_1)$. *Suppose that* ∇f_i *is* L_i*-Lipschitz continuous, and* $r_i \in (0, \frac{2}{L_i})$. *Let* $x_1 \in H_1$, *and* $\mu \in (0, \frac{2\eta}{\kappa^2})$, *and any sequence* $\{\lambda_n\}_{n\in\mathbb{N}} \subset (0, 1]$ *satisfying conditions (E). Let* $\{x_n\}_{n\in\mathbb{N}} \subset H_1$ *be defined by*

$$x_{n+1} = (I - \mu\lambda_n\psi')P_C(I - r_1\nabla f_1)(I - r_2\nabla f_2)\ldots(I - r_\ell\nabla f_\ell)x_n$$

for all $n \in \mathbb{N}$. *Then* $\{x_n\}_{n\in\mathbb{N}}$ *converges strongly to the unique solution* $\bar{x}$ *of problem* $\arg\min_{y\in\bigcap_{i=1}^{m}\arg\min_{x\in H_1}\|A_i x - b_i\|} \psi(y)$.

Proof. Let $T = P_C(I - r_1\nabla f_1)(I - r_2\nabla f_2)\ldots(I - r_\ell\nabla f_\ell)$. For each $i = 1, 2, \ldots, m$, argue as in Theorem 3.4, we see that $I - r_i\nabla f_i$ is averaged, and then T is averaged by Lemma 2.1. Next, by Lemma 2.4, $VI(C, \nabla f_i) = \arg\min_{x\in C} f_1(x)$. Since $Fix(P_C(I - r_i\nabla f_i)) = VI(C, \nabla f_i)$, we know that $\bigcap_{i=1}^{m} \arg\min_{x\in H_1} \|A_i x - b_i\|^2 \neq \emptyset$.

Hence, $\bigcap_{i=1}^{m} FixP_C(I - r_i\nabla f_i)) \neq \emptyset$. By Lemma 2.1,

$$\begin{aligned} Fix(T) &= Fix(P_C)\bigcap\bigcap_{i=1}^{m} Fix(I - r_i\nabla f_i) \\ &= \bigcap_{i=1}^{m} FixP_C(I - r_i\nabla f_i) = \bigcap_{i=1}^{m} \arg\min_{x\in H_1} \|A_i x - b_i\|^2 \neq \emptyset. \end{aligned}$$

Then Theorem 4.8 follows from Theorem 3.1 □

Remark 4.9. The iterations between Theorem 4.8 and [17, Theorem 4.7] are different.

Next, an iteration is used to find the following problem:

$$\arg\min_{x\in D_1} \psi(x), \text{ where } D_1 = \bigcap_{i=1}^{m}\{x \in H_1 : A_i x = b_i\}.$$

Theorem 4.10. *In Theorem 4.8, if condition* $D = \bigcap_{i=1}^{m} \arg\min_{x\in H_1} \|A_i x - b_i\| \neq \emptyset$ *is replaced by* $D_1 = \bigcap_{i=1}^{m}\{x \in H_1 : A_i x = b_i\} \neq \emptyset$. *Then* $\{x_n\}_{n\in\mathbb{N}}$ *converges strongly to the unique solution* $\bar{x}$ *of problem* $\arg\min_{y\in D_1} \psi(y)$.

Proof. Since $D_1 \neq \emptyset$, we know $D \neq \emptyset$. By Theorem 4.8, $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to the unique solution $\bar{x}$ of problem $\arg\min_{y\in\bigcap_{i=1}^{m}\arg\min_{x\in H_1}\|A_i x - b_i\|} \psi(y)$. Since $D_1 \neq \emptyset$, it is easy to see that $D = D_1$ and the conclusion of Theorem 4.10 is true. □

Next, an iteration is used to study the lasso problem.

Theorem 4.11. *Let* A *be a* $m \times n$ *real matrix,* $b \in \mathbb{R}^m$, *and* $t > 0$. *Suppose there exists* $\eta > 0$ *such that* $\|v\|^2\eta \leq \|Av\|^2$ *for all* $v \in H_1$. *Let* $C = \{x \in R^n : \|x\|_1 \leq t\} \neq \emptyset$ *and* $0 < \gamma < \frac{1}{R}$. *Let* $x_1 \in R^n$, $\mu \in (0, \frac{2\eta}{\|A\|^2})$, *and any sequence* $\{\lambda_n\}_{n\in\mathbb{N}} \subset (0, 1]$ *satisfying condition (E). Let* $\{x_n\}_{n\in\mathbb{N}} \subset H_1$ *be defined by*

$$x_{n+1} = P_C(x_n - \mu\lambda_n A^*(AP_C x_n - b))$$

for all $n \in \mathbb{N}$. *Then* $\{x_n\}_{n\in\mathbb{N}}$ *converges strongly to the unique solution of* $\arg\min_{x\in C} \frac{1}{2}\|Ax - b\|_2^2$.

Proof. Let $\psi(x) = \frac{1}{2}\|Ax - b\|_2^2$. Then $\psi'(x) = A^*(Ax - b)$. It is easy to see that $C \subset \mathbb{R}^n$ is a closed convex set. Hence, by Theorem 4.6, $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to the unique solution $\arg\min_{y\in C} \frac{1}{2}\|Ay - b\|_2^2$. □

5. Numerical results

This section mainly focuses on the image deblurring problems, which has received significant attentions in recent years. Several researchers have proposed numerous novel algorithms for this problem based on different deblurring models. For examples, see [6].

All pixels of the original images described in the examples are initially scaled within the range of 0 to 1. The image is subjected to a Gaussian blur measuring 9×9 with a standard deviation 4 (applied using MATLAB functions imfilter and fspecial) followed by an additive zero-mean white Gaussian noise with a standard deviation 10^{-3}. Next, let b be the blurred image, A be the linear operator that corresponds to a spatially invariant point spread function of the blur operation, and $\lambda_n = \frac{1}{\sqrt{n}}$ for all $n \in \mathbb{N}$. Hence, by Theorem 4.11, we get the deblurred image from the blurred image. In fact, the original and observed images are presented in the following figures.

FIGURE 1. Left is the original image, middle is the blurred image from the original image, right is the deblurred image from the blurred image after 2000 times.

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Chih-Sheng Chuang
Department of Applied Mathematics, National Chiayi University, Chiayi, Taiwan
E-mail address: cschuang@mail.ncyu.edu.tw

Lai-Jiu Lin
Department of Mathematics, National Changhua University of Education, Changhua, 50058, Taiwan
E-mail address: maljlin@cc.ncue.edu.tw

Zenn Tsun Yu
Department of Electronic Engineering, Nan Kai University of Technology, Nantou, 542, Taiwan
E-mail address: t106@nkut.edu.tw