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SOLUTIONS FOR MULTIPLE SETS SPLIT FEASIBILITY PROBLEMS

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Delicate to the 70 th birthday of Prof. Wataru Takahashi

Abstract. In this paper, we apply recently result of Lin et al. [16] to study the solution of the following problems: multiple sets split monotone variational inclusion problem, multiple sets split fixed point problem for k-strict pseudo contractive problem, multiple sets split systems of variational inclusion problems, multiple sets split systems of variational inequalities problems, multiple sets split systems of fixed point problem. We give a simple methods to study these problems. Our results contain many original results and will have many applications in many fields of science and mathematics.

1. Introduction

Let $C_1, C_2, \ldots, C_m$ be nonempty closed convex subsets of a real Hilbert space $\mathcal{H}$. The well-known convex feasibility problem **(CFP)** is to find $x^* \in \mathcal{H}$ such that

$$x^* \in C_1 \cap C_2 \cap \cdots \cap C_m.$$

Convex feasibility problem has received a lot of attention due to its diverse applications in mathematics, approximation theory, communications, geophysics, control theory, biomedical engineering. One can refer to [10, 22].

The split feasibility problem **(SFP)** is to find a point

$$x^* \in C \text{ such that } Ax^* \in Q,$$

where C, Q are nonempty closed convex subsets of real Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$, respectively. $A : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator. The split feasibility problem (**SFP**) in finite dimensional real Hilbert spaces was first introduced by Censor and Elfving [7] for modeling inverse problems which arise from medical image reconstruction. Since then, the split feasibility problem (**SFP**) has received much attention due to its applications in signal processing, image reconstruction, approximation theory, control theory, biomedical engineering, communications, and geophysics. For examples, one can refer to [1,5–10,13–15,17,18,22,23,26] and related literatures.

In 2011, Moudafi [19] introduced and studied the following split monotone variational inclusion (**SMVI**) :

$$\text{Find } \bar{x} \in H_1 \text{ such that } \bar{x} \in (B_1 + G_1)^{-1}0, \tag{1.1}$$

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and

$$\bar{y} = A\bar{x} \in H_2 \text{ such that } \bar{y} \in (B+G)^{-1}0, \tag{1.2}$$

where H_1 and H_2 are real Hilbert spaces, $A : H_1 \to H_2$ is a bounded linear operator, $B_1 : H_1 \to H_1$ and $B : H_2 \to H_2$ are given operators, $G_1 : H_1 \multimap H_1$ and $G : H_2 \multimap H_2$ are given multivalued mappings.

Moudafi [19] proved the following weakly convergence theorem for the solution of the split monotone variational inclusion (**SMVI**) with the iteration defined by

$$\begin{cases} x_1 \in C \text{ chosen arbitrarily,} \\ x_{n+1} = J_\lambda^{G_1}(I-\lambda B_1)(I-\gamma A^*(I-T)A)x_n, \end{cases}$$

where $J_\lambda^{G_1}$ is the resolvent of G_1 defined by $J_\lambda^{G_1} = (I+\lambda G_1)^{-1}$ for each $\lambda > 0$ and $T = J_r^G(I - rB)$ for each $r > 0$.

Let $C_1, C_2, \ldots, C_m$ be nonempty closed convex subsets of $\mathcal{H}_1$, $Q_1, Q_2, \ldots, Q_m$ be nonempty closed convex subsets of $\mathcal{H}_2$, and $A_1, A_2, \ldots, A_m : \mathcal{H}_1 \to \mathcal{H}_2$ be bounded linear operators. The well-known multiple sets split feasibility problem **(MSSFP)** is to find $x^* \in \mathcal{H}_1$ such that

$$x^* \in C_i \text{ such that } A_i x^* \in Q_i \text{ for all } i = 1, 2, \ldots, m.$$

Motivated by the above works, recently Lin et al. [16] considered the following algorithm:

$$\begin{cases} v_0 \in \mathcal{H} \text{ is chosen arbitrarily,} \\ v_{2n+1} := a_n u + b_n v_{2n} + c_n J_{\delta_n}^{G_1}(I-\delta_n B_1)v_{2n}, \ n \in \mathbb{N} \cup \{0\}, \\ v_{2n} := f_n u + g_n v_{2n-1} + h_n J_{\gamma_n}^{G_2}(I-\gamma_n B_2)v_{2n-1}, \ n \in \mathbb{N}, \end{cases} \tag{1.3}$$

where G_1, G_2 are two set-valued maximal monotone mappings on a real Hilbert space $\mathcal{H}_1$, $B_1, B_2 : C \to \mathcal{H}_1$ are two mappings, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{f_n\}$, $\{g_n\}$ and $\{h_n\}$ are sequences in $[0, 1]$. Lin et al. [16] showed the sequence $\{v_n\}$ generated by (1.3) converges strongly to some $\bar{x} \in (B_1+G_1)^{-1}(0) \cap (B_2+G_2)^{-1}(0)$ under suitable conditions.

In this paper, we apply recently result of Lin et al. [16] to study the solution of the following problems: multiple sets split monotone variational inclusion problem, multiple sets split fixed point problem for k-strict pseudo contractive problem, multiple sets split systems of variational inclusion problems, multiple sets split systems of variational inequalities problems, multiple sets split systems of fixed point problem. We give a simple methods to study these problems. Our results contain many original results and will have many applications in many fields of science and mathematics.

2. Preliminaries

Throughout this paper, let $\mathcal{H}$, $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$ denote the real Hilbert spaces with inner product $\langle .,. \rangle$ and norm $|| \cdot ||$, $\mathbb{N}$ the set of all natural numbers, and $\mathbb{R}^+$ be the set of all positive real numbers. A set-valued mapping A with domain $\mathcal{D}(A)$ on $\mathcal{H}$ is called monotone if $\langle u - v, x - y \rangle \geq 0$ for any $u \in Ax$, $v \in Ay$ and for all x, $y \in \mathcal{D}(A)$. A monotone operator A is called maximal monotone if its graph $\{(x, y) : x \in \mathcal{D}(A), y \in Ax\}$ is not properly contained in the graph of any other monotone mapping. The set of all zero points of A is denoted by $A^{-1}(0)$, i.e.,

$A^{-1}(0) = \{x \in \mathcal{H} : 0 \in Ax\}$. In what follows, we denote the strongly convergence and the weak convergence of $\{x_n\}$ to $x \in \mathcal{H}$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. In order to facilitate our discussion in the next section, we recall some facts. The following equality is easy to check:

(2.1)

$$||\alpha x+\beta y+\gamma z||^2 = \alpha||x||^2+\beta||y||^2+\gamma||z||^2-\alpha\beta||x-y||^2-\alpha\gamma||x-z||^2-\beta\gamma||y-z||^2$$

for each $x, y, z \in \mathcal{H}$ and $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha + \beta + \gamma = 1$. Besides, we also have

$$||x + y||^2 \leq ||x||^2 + 2\langle y, x + y\rangle \tag{2.2}$$

for each $x, y \in \mathcal{H}$. Let C be a nonempty closed convex subset of $\mathcal{H}$, and a mapping $T : C \to \mathcal{H}$. We denote the set of all fixed points of T by $Fix(T)$. A mapping $T : C \to \mathcal{H}$ is said to be nonexpansive if $||Tx - Ty|| \leq ||x - y||$ for every $x, y \in C$. A mapping $T : C \to \mathcal{H}$ is said to be quasi-nonexpansive if $Fix(T) \neq \emptyset$ and $||Tx-y|| \leq ||x - y||$ for all $x \in C$ and $y \in Fix(T)$. A mapping $T : C \to \mathcal{H}$ is said to be firmly nonexpansive if

$$||Tx - Ty||^2 \leq ||x - y||^2 - ||(I - T)x - (I - T)y||^2$$

for every $x, y \in C$. Besides, it is easy to see that $Fix(T)$ is a closed convex subset of C if $T : C \to \mathcal{H}$ is a quasi-nonexpansive mapping. A mapping $T : C \to \mathcal{H}$ is said to be α-inverse-strongly monotone (α-ism) if

$$\langle x - y, Tx - Ty\rangle \geq \alpha\|Tx - Ty\|^2$$

for all $x, y \in \mathcal{H}$ and $\alpha > 0$.

The follows lemmas are needed in this paper.

Lemma 2.1 ([29]). *Let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator, and A^* the adjoint of A. Suppose that C is a nonempty closed convex subset of $\mathcal{H}_2$, and $F : C \to \mathcal{H}_2$ is a firmly nonexpansive mapping. Then $A^*(I - F)A$ is a $\frac{1}{\|A\|^2}$-ism, that is,*

$$\frac{1}{\|A\|^2}||A^*(I - F)Ax - A^*(I - F)Ay||^2 \leq \langle x - y, A^*(I - F)Ax - A^*(I - F)Ay\rangle$$

for all $x, y \in \mathcal{H}_1$.

Lemma 2.2 ([2]). *Let C be a nonempty closed convex subset of $\mathcal{H}$, and $F : C \to \mathcal{H}$ a firmly nonexpansive mapping. Suppose that $Fix(F)$ is nonempty. Then $\langle x - Fx, Fx - w\rangle \geq 0$ for each $x \in \mathcal{H}$ and each $w \in Fix(F)$.*

Let C be a nonempty closed convex subset of $\mathcal{H}$. Then for each $x \in \mathcal{H}$, there is a unique element $\bar{x} \in C$ such that $||x - \bar{x}|| = \min_{y\in C} ||x - y||$. Here, we set $P_C x = \bar{x}$ and P_C is said to be the metric projection from $\mathcal{H}$ onto C.

Lemma 2.3 ([25]). *Let C be a nonempty closed convex subset of $\mathcal{H}$, and let P_C be the metric projection from $\mathcal{H}$ onto C. Then $\langle x - P_C x, P_C x - y\rangle \geq 0$ for each $x \in \mathcal{H}$ and each $y \in C$.*

For a set-valued maximal monotone operator G on $\mathcal{H}$ and $r > 0$, we may define an operator $J_r^G : \mathcal{H} \to \mathcal{H}$ with $J_r^G = (I + rG)^{-1}$ which is called the resolvent mapping of G for r.

A mapping $T : \mathcal{H} \to \mathcal{H}$ is said to be averaged if $T = (1-\alpha)I + \alpha S$, where $\alpha \in (0,1)$ and $S : \mathcal{H} \to \mathcal{H}$ is a nonexpansive mapping.

Lemma 2.4 ([11]). *Let C be a nonempty closed convex subset of $\mathcal{H}$, and $T : C \to \mathcal{H}$ a mapping. Then the following hold:*

(i) *T is nonexpansive mapping if and only if $I-T$ is $\frac{1}{2}$-inverse-strongly monotone ($\frac{1}{2}$-ism).*
(ii) *If S is ν-ism, then γS is $\frac{\nu}{\gamma}$-ism.*
(iii) *S is averaged if and only if $I-S$ is ν-ism for some $\nu > \frac{1}{2}$. Indeed, S is α-averaged if and only if $I-S$ is $\frac{1}{(2\alpha)}$-ism, for $\alpha \in (0,1)$.*
(iv) *If S and T are averaged, then the composition ST is also averaged.*
(v) *If the mappings $\{T_i\}_{i=1}^n$ are averaged and have a common fixed point, then $\bigcap_{i=1}^n Fix(T_i) = Fix(T_1 \cdots T_n)$ for each $n \in \mathbb{N}$.*

Let C be a nonempty closed convex subset of $\mathcal{H}$. The indicator function ι_C defined by

$$\iota_C x = \begin{cases} 0, \ x \in C \\ \emptyset, \ x \notin C \end{cases}$$

is a proper lower semicontinuous convex function and its subdifferential $\partial\iota_C$ defined by

$$\partial\iota_C x = \{z \in \mathcal{H} : \langle y-x, z\rangle \le \iota_C(y) - \iota_C(x), \ \forall y \in \mathcal{H}\}$$

is a maximal monotone operator [21]. Furthermore, we also define the normal cone $N_C u$ of C at u as follows;

$$N_C u = \{z \in \mathcal{H} : \langle z, v-u\rangle \le 0, \ \forall v \in C\}.$$

We can define the resolvent $J_\lambda^{\partial i_C}$ of ∂i_C for $\lambda > 0$, i.e.

$$J_\lambda^{\partial i_C} x = (I + \lambda\partial i_C)^{-1} x$$

for all $x \in \mathcal{H}$. Since

$$\begin{aligned} \partial i_C x &= \{z \in \mathcal{H} : i_C x + \langle z, y-x\rangle \le i_C y, \ \forall y \in \mathcal{H}\} \\ &= \{z \in \mathcal{H} : \langle z, y-x\rangle \le 0, \ \forall y \in C\} \\ &= N_C x \end{aligned}$$

for all $x \in C$, we have that

$$\begin{aligned} u = J_\lambda^{\partial i_C} x \ &\Leftrightarrow x \in u + \lambda\partial i_C u \\ &\Leftrightarrow x - u \in \lambda N_C u \\ &\Leftrightarrow \langle x-u, y-u\rangle \le 0, \ \forall y \in C \\ &\Leftrightarrow u = P_C x. \end{aligned}$$

Let C, Q and Q' be nonempty closed convex subsets of $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$, respectively. For each $i = 1, 2$, and $\kappa_i > 0$, let B_i be a κ_i–inverse-strongly monotone mapping of C into $\mathcal{H}_1$, G_i a set-valued maximal monotone mapping on $\mathcal{H}_1$ such that the domain of G_i is included in C. Let G be a set-valued maximal monotone mapping on $\mathcal{H}_2$ such that the domain of G is included in Q and let G' be a set-valued maximal monotone mapping on $\mathcal{H}_3$ such that the domain of G' is included in Q'. For

$\nu > 0$, let B be a $\nu-$inverse-strongly monotone mapping of Q into $\mathcal{H}_2$. For $\nu^{'} > 0$, let $B^{'}$ be a $\nu^{'}-$inverse-strongly monotone mapping of $Q^{'}$ into $\mathcal{H}_3$. Let F_1 be a firmly nonexpansive mapping of $\mathcal{H}_2$ into $\mathcal{H}_2$ and F_2 a firmly nonexpansive mapping of $\mathcal{H}_3$ into $\mathcal{H}_3$. Let T_i be an averaged mappings of $\mathcal{H}_2$ into $\mathcal{H}_2$ for $i = 1, 2 \ldots, m$ and S_j be an averaged mapping of $\mathcal{H}_3$ into $\mathcal{H}_3$ for $j = 1, 2, \ldots, n$. Note $J_\lambda^{G_1} = (I + \lambda G_1)^{-1}$ and $J_r^{G_2} = (I + rG_2)^{-1}$ for each $\lambda > 0$ and $r > 0$. Let $A_1 : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator, $A_2 : \mathcal{H}_1 \to \mathcal{H}_3$ a bounded linear operator, and A_i^* be the adjoint of A_i for $i = 1, 2$. Throughout this paper, we use these notations unless specified otherwise.

Theorem 2.5 ([16]). *Suppose that* $(B_1+G_1)^{-1}(0)\cap(B_2+G_2)^{-1}(0)$ *is nonempty, and* $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{f_n\}$, $\{g_n\}$, $\{h_n\}$ *are sequences in* $[0,1]$ *such that* $a_n+b_n+c_n = 1$, $f_n + g_n + h_n = 1$, $0 < a_n < 1$, *and* $0 < f_n < 1$ *for each* $n \in \mathbb{N}$. *For an arbitrarily fixed* $u \in \mathcal{H}$. *Define a sequence* $\{v_n\}$ *by*

$$\begin{cases} v_0 \in \mathcal{H} \text{ is chosen arbitrarily,} \\ v_{2n+1} := a_n u + b_n v_{2n} + c_n J_{\delta_n}^{G_1}(I - \delta_n B_1)v_{2n}, \ n \in \mathbb{N}\cup\{0\}, \\ v_{2n} := f_n u + g_n v_{2n-1} + h_n J_{\gamma_n}^{G_2}(I - \gamma_n B_2)v_{2n-1}, \ n \in \mathbb{N}. \end{cases} \tag{2.3}$$

Then $\lim_{n\to\infty} v_n = P_{(B_1+G_1)^{-1}(0)\cap(B_2+G_2)^{-1}(0)}u$ *provided the following conditions are satisfied:*

(i) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} f_n = 0$;
(ii) *either* $\sum_{n=1}^{\infty} a_n = \infty$ *or* $\sum_{n=1}^{\infty} f_n = \infty$;
(iii) $\delta_n \subset (0,\infty)$, $\gamma_n \subset (0,\infty)$, $0 < a \le \delta_n \le b < 2\kappa_1$ *and* $0 < f \le \gamma_n \le g < 2\kappa_2$, *for each* $n \in \mathbb{N}$ *and for some* $a, b, f, g \in \mathbb{R}^+$;
(iv) $\liminf_{n\to\infty} c_n > 0$, $\liminf_{n\to\infty} h_n > 0$.

3. Multiple sets split fixed point problem

Let C be a nonempty closed convex subset of $\mathcal{H}$. Let $g : C \times C \to \mathbb{R}$. Then the equilibrium problem is to find $\hat{x} \in C$ such that

$$g(\hat{x}, y) \ge 0, \text{ for all } y \in C$$

whose solution set is denoted by $EP(g)$. For solving an equilibrium problem, we may assume the bifunction g satisfies the following conditions such that
(A1) $g(x,x) = 0, \ \forall \ x \in C$;
(A2) g is monotone, that is, $g(x,y) + g(y,x) \le 0, \ \forall \ x \in C$;
(A3) for all $x, y, z \in C, \limsup_{t\downarrow 0} g((1-t)x + tz, y) \le g(x,y)$;
(A4) for all $x \in C$, $g(x,\cdot)$ is convex and lower semicontinuous.

We have the following lemmas from Blum and Oettli [3], and Combettes and Hirstoaga [12].

Lemma 3.1 ([3]). *Let* C *be a nonempty closed convex subset of* $\mathcal{H}$ *and* $g : C\times C \to \mathbb{R}$ *a function satisfying conditions (A1)–(A4), and suppose* $r > 0$, $x \in \mathcal{H}$. *Then, there exists a unique* $z \in C$ *such that*

$$g(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \ \textit{for all } y \in C.$$

Lemma 3.2 ([12]). *Let C be a nonempty closed convex subset of $\mathcal{H}$ and $g : C \times C \to \mathbb{R}$ a function satisfying conditions (A1)–(A4). For $r > 0$, define $J_r^g : \mathcal{H} \to C$ by*

$$J_r^g x = \left\{ z \in C : g(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \ \forall \ y \in C \right\}.$$

for all $x \in \mathcal{H}$. Then the following hold:
(a) *J_r^g is single-valued;*
(b) *J_r^g is firmly nonexpansive;*
(c) *$Fix(J_r^g) = EP(g)$;*
(d) *$EP(g)$ is closed and convex.*

We call J_r^g the resolvent of g for $r > 0$.
Takahashi, Takahashi and Toyoda [24] gave the following lemma.

Lemma 3.3 ([24]). *Let $g : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions (A1)-(A4). Define A_g as follows:*

$$A_g x = \begin{cases} \{z \in H_1 : g(x, y) \geq \langle y - x, z \rangle, \forall y \in C\} & \text{if} \quad x \in C; \\ \emptyset & \text{if} \quad x \notin C. \end{cases} \tag{3.1}$$

Then, $EP(g) = A_g^{-1}0$ and A_g is a maximal monotone operator with the domain of $A_g \subset C$. Furthermore, for any $x \in H_1$ and $r > 0$, the resolvent T_r^g of g coincides with the resolvent of A_g, i.e., $T_r^g x = (I + rA_g)^{-1}x$.

Recently Yu et al. [28] give an essential result in this paper for the following essential problem (**SFP-1**):
Find $\bar{x} \in H_1$ such that $A_1\bar{x} \in (B + G)^{-1}(0)$.

Lemma 3.4 ([28]). *Given any $\bar{x} \in H_1$*
(i) *If $\bar{x}$ is a solution of $(\mathbf{SPF-1})$, then $(I - \lambda A_1^*(I - U_1)A_1)\bar{x} = \bar{x}$ where $U_1 = J_\sigma^G(I - \sigma B)$.*
(ii) *Suppose that $U_1 = J_\sigma^G(I - \sigma B)$, $0 < \lambda < \frac{1}{R_1}$, $0 < \sigma < 2\nu$. Then $A_1^*(I - U_1)A_1$ is a $\frac{\mu_1}{R_1}$-ism mapping, $J_\sigma^G(I - \sigma B)$ and $(I - \lambda A_1^*(I - U_1)A_1)$ are averaged.*
(iii) *Suppose that $U_1 = J_\sigma^G(I - \sigma B)$, $0 < \lambda < \frac{1}{R_1}$, $0 < \sigma < 2\nu$, $(I - \lambda A_1^*(I - U_1)A_1)\bar{x} = \bar{x}$ and the solutions set of $(\mathbf{SPF-1})$ is nonempty. Then $\bar{x}$ is a solution of $(\mathbf{SPF-1})$,*

The following lemma whose proof is essential the same as Theorem 4.1 in [28] is a special case of Theorem 3.2 in [27]:

(**SFP-2**) Find $\bar{x} \in \mathcal{H}_1$ such that $A_1\bar{x} \in Fix(F_1)$.

Lemma 3.5. *Given any $\bar{x} \in \mathcal{H}_1$*
(i) *If $\bar{x}$ is a solution of $(\mathbf{SFP-2})$, then $(I - \rho_n A_1^*(I - F_1)A_1)\bar{x} = \bar{x}$ for each $n \in \mathbb{N}$.*
(ii) *Suppose that $0 < \rho_n < \frac{2}{\|A_1\|^2+2}$, for each $n \in \mathbb{N}$. Then $A_1^*(I - F_1)A_1$ is a $\frac{1}{\|A_1\|^2}$-ism mapping and $(I - \rho_n A_1^*(I - F_1)A_1)$ are averaged. Suppose further that $(I - \rho_n A_1^*(I - F_1)A_1)\bar{x} = \bar{x}$ and the solution set of $(\mathbf{SFP-2})$ is nonempty. Then $\bar{x}$ is a solution of $(\mathbf{SFP-2})$.*

Recently, Lin et al. [16] study the following problem.

(**SFP–3**) Find $\bar{x} \in \mathcal{H}_1$ such that $\bar{x} \in Fix(J_{\rho_n}^{G_1})$ and $A_1\bar{x} \in Fix(F_1)$.

As a special case of Lemma 3.5, we have the following recently result of Lin et al. [16].

Lemma 3.6 ([16]). *Given any $\bar{x} \in \mathcal{H}_1$.*

(i) *If $\bar{x}$ is a solution of* (**SFP − 3**)*, then $J_{\rho_n}^{G_1}(I - \rho_n A_1^*(I - F_1)A_1)\bar{x} = \bar{x}$ for each $n \in \mathbb{N}$.*

(ii) *Suppose that $J_{\rho_n}^{G_1}(I - \rho_n A_1^*(I - F_1)A_1)\bar{x} = \bar{x}$ with $0 < \rho_n < \frac{2}{\|A_1\|^2+2}$, for each $n \in \mathbb{N}$ and the solution set of* (**SFP − 3**) *is nonempty. Then $(I - \rho_n A_1^*(I - F_1)A_1)$ and $J_{\rho_n}^{G_1}(I - \rho_n A_1^*(I - F_1)A_1)$ are averaged and $\bar{x}$ is a solution of* (**SFP − 3**).

Proof. To prove(ii), suppose that all the assumption is satisfied. Then there exists $w \in \mathcal{H}_1$ such that $w \in Fix(J_{\rho_n}^{G_1})$ and $A_1 w \in Fix(F_1)$. Hence $w \in Fix(J_{\rho_n}^{G_1}) \bigcap Fix(I - \rho_n A_1^*(I - F_1)A_1)$. Since $J_{\rho_n}^{G_1}(I - \rho_n A_1^*(I - F_1)A_1)\bar{x} = \bar{x}$, it follows from Lemma 2.4 that $\bar{x} \in Fix(J_{\rho_n}^{G_1}) \bigcap Fix(I - \rho_n A_1^*(I - F_1)A_1)$. Therefore $\bar{x} \in Fix(J_{\rho_n}^{G_1})$ and $\bar{x} \in Fix(I - \rho_n A_1^*(I - F_1)A_1)$. Then Lemma 3.6 follows from Lemma 3.5. □

Now, we recall the following multiple sets split feasible problem (**MSSFP-A1**):

$$\begin{cases} \text{Find } \bar{x} \in H_1 \text{ such that } \bar{x} \in G_1^{-1}(0) \cap G_2^{-1}(0), \\ A_1\bar{x} \in Fix(F_1) \text{ and } A_2\bar{x} \in Fix(F_2). \end{cases}$$

Theorem 3.7 ([16]). *Suppose that the solutions set Ω_{A1} of* (**MSSFP − A1**) *is nonempty, and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{f_n\}$, $\{g_n\}$, $\{h_n\}$ are sequences in $[0, 1]$ with $a_n + b_n + c_n = 1$, $f_n + g_n + h_n = 1$, $0 < a_n < 1$, and $0 < f_n < 1$ for each $n \in \mathbb{N}$. For an arbitrary fixed $u \in \mathcal{H}$, a sequence $\{v_n\}$ be defined by*

$$\begin{cases} v_{2n+1} := & a_n u + b_n v_{2n} + c_n J_{\rho_n}^{G_1}(I - \rho_n A_1^*(I - F_1)A_1)v_{2n}, \ n \in \mathbb{N} \cup \{0\}, \\ v_{2n} := & f_n u + g_n v_{2n-1} + h_n J_{\sigma_n}^{G_2}(I - \sigma_n A_2^*(I - F_2)A_2)v_{2n-1}, \ n \in \mathbb{N}. \end{cases}$$

Then $\lim_{n\to\infty} v_n = P_{\Omega_{A1}} u$ provided the following conditions are satisfied:

(i) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} f_n = 0$;

(ii) *either* $\sum_{n=1}^{\infty} a_n = \infty$ *or* $\sum_{n=1}^{\infty} f_n = \infty$;

(iii) $0 < \rho_n < \frac{2}{\|A_1\|^2+2}$, $0 < \sigma_n < \frac{2}{\|A_2\|^2+2}$ *for each* $n \in \mathbb{N}$;

(iv) $\liminf_{n\to\infty} c_n > 0$ *and* $\liminf_{n\to\infty} h_n > 0$.

We consider the following multiple sets split monotonic variational inclusion problem (**MSSMVIP − B1**):

$$\begin{cases} \text{Find } \bar{x} \in H_1 \text{ such that } \bar{x} \in G_1^{-1}(0) \cap G_2^{-1}(0), \\ A_1\bar{x} \in (B + G)^{-1}(0) \text{ and } A_2\bar{x} \in (B' + G')^{-1}(0). \end{cases}$$

That is,

$$\begin{cases} \text{Find } \bar{x} \in H_1 \text{ such that } \bar{x} \in Fix(J_\lambda^{G_1}) \bigcap Fix(J_r^{G_2}), \\ A_1\bar{x} \in Fix(U_1) \text{ and } A_2\bar{x} \in Fix(U_2) \text{ where} \\ \quad U_1 = J_\sigma^G(I - \sigma B), U_2 = J_{\sigma'}^{G'}(I - \sigma' B'). \end{cases}$$

Let Ω_{B1} be the solutions set of (**MSSMVIP − B1**).

Theorem 3.8. *Suppose that the solutions set Ω_{B1} of* (**MSSFP − B1**) *is nonempty, and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{f_n\}$, $\{g_n\}$, $\{h_n\}$ are sequences in $[0,1]$ with $a_n+b_n+c_n=1$, $f_n+g_n+h_n=1$, $0<a_n<1$, and $0<f_n<1$ for each $n\in\mathbb{N}$. For an arbitrary fixed $u\in\mathcal{H}$, a sequence sequence $\{v_n\}$ is defined by*

$$\begin{cases} v_{2n+1} := & a_n u + b_n v_{2n} + c_n J_\lambda^{G_1}(I-\lambda A_1^*(I-U_1)A_1)v_{2n},\ n\in\mathbb{N}\cup\{0\}, \\ v_{2n} := & f_n u + g_n v_{2n-1} + h_n J_r^{G_2}(I-rA_2^*(I-U_2)A_2)v_{2n-1},\ n\in\mathbb{N}, \end{cases}$$

where $U_1=J_\sigma^G(I-\sigma B), U_2=J_{\sigma'}^{G'}(I-\sigma' B')$. Then $\lim_{n\to\infty} v_n = P_{\Omega_{B1}}u$. provided the following conditions are satisfied:

(i) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} f_n = 0$;

(ii) *either* $\sum_{n=1}^{\infty} a_n = \infty$ *or* $\sum_{n=1}^{\infty} f_n = \infty$;

(iii) $0<\lambda<\frac{1}{R_1}$, $0<r<\frac{1}{R_2}$, $0<\sigma<2\nu$ *and* $0<\sigma'<2\nu'$;

(iv) $\liminf_{n\to\infty} c_n > 0$ *and* $\liminf_{n\to\infty} h_n > 0$.

Proof. For each $i=1,2$, by Lemma 3.4, $A_i^*(I-U_i)A_i$ is $\frac{\mu_i}{R_i}$-ism for some $\mu_i>\frac{1}{2}$. Put $B_1=A_1^*(I-U_1)A_1$ and $B_2=A_2^*(I-U_2)A_2$ in Theorem 2.5. Then algorithm in Theorem 2.5 follows immediately from algorithm in Theorem 3.8.
Since the solution set of (**MSSFP − B1**) is nonempty, we have,
(3.2)

$$\bar{w}\in FixJ_\lambda^{G_1}\bigcap Fix(I-\lambda A_1^*(I-U_1)A_1))\bigcap FixJ_r^{G_2}\bigcap Fix(I-rA_2^*(I-U_2)A_2)\neq\emptyset.$$

This implies that,

$$\bar{w}\in Fix(J_\lambda^{G_1}(I-\lambda B_1))\bigcap Fix(J_r^{G_2}(I-rB_2))\neq\emptyset$$

and

$$\bar{w}\in(B_1+G_1)^{-1}0\bigcap(B_2+G_2)^{-1}0\neq\emptyset.$$

By Theorem 2.5, $\lim_{n\to\infty} v_n = P_{(B_1+G_1)^{-1}0\cap(B_2+G_2)^{-1}0}u$.
On the other hand, by Lemma 3.4, we have that

(3.3) $I-\lambda B_1 = I-\lambda A_1^*(I-U_1)A_1$ and $I-rB_2=I-rA_2^*(I-U_2)A_2$ are averaged .

since $J_\lambda^{G_1}$, $J_r^{G_2}$ are firmly nonexpansive mappings, it easy see that

$$J_\lambda^{G_1},\ J_r^{G_2} \text{ are } \frac{1}{2} \text{ averaged.} \tag{3.4}$$

By (3.2), (3.3), (3.4) and Lemma 2.4(v), we see that

$$\begin{aligned} &Fix(J_\lambda^{G_1})\bigcap Fix(I-\lambda B_1)\bigcap Fix(J_r^{G_2})\bigcap Fix(I-rB_2) \\ &= Fix(J_\lambda^{G_1}(I-\lambda B_1))\bigcap Fix(J_r^{G_2}(I-rB_2)). \end{aligned} \tag{3.5}$$

If $w\in(B_1+G_1)^{-1}0\cap(B_2+G_2)^{-1}0$, we have that $w\in Fix(J_\lambda^{G_1}(I-\lambda B_1))\bigcap Fix(J_r^{G_2}(I-rB_2))$. By (3.5), we have that
(3.6)

$$\begin{aligned} &w\in Fix(J_\lambda^{G_1})\bigcap Fix(I-\lambda B_1)\bigcap Fix(J_r^{G_2})\bigcap Fix(I-rB_2) \\ &= Fix(J_\lambda^{G_1})\bigcap Fix(I-\lambda A_1^*(I-U_1)A_1))\bigcap Fix(J_r^{G_2})\bigcap Fix(I-rA_2^*(I-U_2)A_2). \end{aligned}$$

By (3.6) and Lemma 3.4(iii), $w \in G_1^{-1}0 \cap G_2^{-1}0$, $A_1w \in (B+G)^{-1}0$ and $A_2w \in (B'+G')^{-1}0$. Therefore, $(B_1+G_1)^{-1}0 \cap (B_2+G_2)^{-1}0 \subseteq \Omega_{B1}$. If $w \in \Omega_{B1}$, we have that

(3.7)

$$\begin{aligned} &w \in FixJ_\lambda^{G_1} \bigcap Fix(I-\lambda A_1^*(I-U_1)A_1)) \bigcap FixJ_r^{G_2} \bigcap Fix(I-rA_2^*(I-U_2)A_2) \\ &= FixJ_\lambda^{G_1} \bigcap Fix(I-\lambda B_1) \bigcap FixJ_r^{G_2} \bigcap Fix(I-rB_2). \end{aligned}$$

This implies that $w \in FixJ_\lambda^{G_1}(I-\lambda B_1) \bigcap FixJ_r^{G_2}(I-rB_2) = (B_1+G_1)^{-1}0 \cap (B_2+G_2)^{-1}0$. Therefore, $\Omega_{B1} \subseteq (B_1+G_1)^{-1}0 \cap (B_2+G_2)^{-1}0$ and $(B_1+G_1)^{-1}0 \cap (B_2+G_2)^{-1}0 = \Omega_{B1}$. This complete the proof of Theorem 3.8. □

Remark 3.9. The proof, iteration of Theorem 3.8 are different from Theorem 4.2 [28]. In Theorem 4.2 [28], we use a result of hierarchical inequality to study the problem $(\mathbf{MSSMVIP-B1})$, but in theorem 3.8, we use proximal point algorithm to study this problem. Theorem 3.8 improves Theorem 3.1 [19].

Now, we consider the following multiple sets split monotonic variational inclusion problem $(\mathbf{MSSMVIP-C1})$:

$$\begin{cases} \text{Find } \bar{x} \in H_1 \text{ such that } \bar{x} \in G_1^{-1}(0) \cap G_2^{-1}(0), \\ A_1\bar{x} \in Fix(F_1) \text{ and } A_2\bar{x} \in (B'+G')^{-1}(0). \end{cases}$$

That is,

$$\begin{cases} \text{Find } \bar{x} \in H_1 \text{ such that } \bar{x} \in Fix(J_\lambda^{G_1}) \bigcap Fix(J_r^{G_2}), \\ A_1\bar{x} \in Fix(F_1) \text{ and } A_2\bar{x} \in Fix(U_2) \text{ where } U_2 = J_{\sigma'}^{G'}(I-\sigma' B'). \end{cases}$$

Let Ω_{C1} be the solutions set of $(\mathbf{MSSMVIP-B1})$.

Theorem 3.10. *Suppose that the solutions set Ω_{C1} of $(\mathbf{MSSFP-C1})$ is nonempty, and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{f_n\}$, $\{g_n\}$, $\{h_n\}$ are sequences in $[0,1]$ with $a_n+b_n+c_n=1$, $f_n+g_n+h_n=1$, $0<a_n<1$, and $0<f_n<1$ for each $n \in \mathbb{N}$. For an arbitrary fixed $u \in \mathcal{H}$, a sequence sequence $\{v_n\}$ is defined by*

$$\begin{cases} v_{2n+1} := & a_n u + b_n v_{2n} + c_n J_{\rho_n}^{G_1}(I-\rho_n A_1^*(I-F_1)A_1)v_{2n}, \ n \in \mathbb{N}\cup\{0\}, \\ v_{2n} := & f_n u + g_n v_{2n-1} + h_n J_r^{G_2}(I-rA_2^*(I-U_2)A_2)v_{2n-1}, \ n \in \mathbb{N}, \end{cases}$$

where $U_2 = J_{\sigma'}^{G'}(I-\sigma' B')$. Then $\lim_{n\to\infty} v_n = P_{\Omega_{C1}}u$ provided the following conditions are satisfied:

(i) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} f_n = 0$;
(ii) *either* $\sum_{n=1}^\infty a_n = \infty$ *or* $\sum_{n=1}^\infty f_n = \infty$;
(iii) $0 < \rho_n < \frac{2}{\|A_1\|^2+2}$, $0 < r < \frac{1}{R_2}$ *and* $0 < \sigma' < 2\nu'$ *for each* $n \in \mathbb{N}$;
(iv) $\liminf_{n\to\infty} c_n > 0$, $\liminf_{n\to\infty} h_n > 0$.

Proof. Since F_i is a firmly nonexpansive, it follow from Lemma 3.5 that we have that $A_1^*(I-F_1)A_1 : C_1 \to H_1$ is $\frac{1}{\|A_1\|^2}$-ism. By Lemma 3.4, $A_2^*(I-U_2)A_i$ is $\frac{\mu_2}{R_2}$-ism. Put $B_1 = A_1^*(I-F_1)A_1$ and $B_2 = A_2^*(I-U_2)A_2$ in Theorem 2.5. Then algorithm in Theorem 2.5 follows immediately from algorithm in Theorem 3.10.

Since the solution set of $(\mathbf{MSSFP-C1})$ is nonempty, by Lemmas 3.4 and 3.5, we have that, $\bar{w} \in Fix(J_\lambda^{G_1}(I-\lambda A_1^*(I-F_1)A_1)) \bigcap Fix(J_r^{G_2}(I-rA_2^*(I-U_2)A_2) \neq \emptyset$.

This implies that, $\bar{w} \in (B_1 + G_1)^{-1}0 \bigcap (B_2 + G_2)^{-1}0 \neq \emptyset$.

By Theorem 2.5, $\lim_{n\to\infty} v_n = P_{(B_1+G_1)^{-1}(0)\cap(B_2+G_2)^{-1}(0)}u$. Let $\lim_{n\to\infty} v_n = \bar{x}$. Then follow the same arguments as in Theorems 3.7 and 3.8. We can prove Theorem 3.10. □

4. Multiple sets split system of variational inequalities problems

Let T be a nonexpansive mappings of $\mathcal{H}_2$ into $\mathcal{H}_2$ and S be a nonexpansive mapping of $\mathcal{H}_3$ into $\mathcal{H}_3$.

Now, we study the following multiple sets split feasible problem (**MSSFP–D1**):

Find $\bar{x} \in \mathcal{H}_1$ such that $\bar{x} \in G_1^{-1}(0) \cap G_2^{-1}(0)$, $A_1\bar{x} \in Fix(T)$ and $A_2\bar{x} \in Fix(S)$.

Theorem 4.1. *Suppose that the solutions set Ω_{D1} of $(\mathbf{MSSFP-D1})$ is nonempty, and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{f_n\}$, $\{g_n\}$, $\{h_n\}$ are sequences in $[0,1]$ with $a_n+b_n+c_n = 1$, $f_n + g_n + h_n = 1$, $0 < a_n < 1$, and $0 < f_n < 1$ for each $n \in \mathbb{N}$. For an arbitrary fixed $u \in \mathcal{H}$, a sequence $\{v_n\}$ be defined by*

$$\begin{cases} v_{2n+1} := & a_nu + b_nv_{2n} + c_nJ^{G_1}_{\rho_n}(I - \rho_nA_1^*(I - T)A_1)v_{2n}, \ n \in \mathbb{N} \cup \{0\}, \\ v_{2n} := & f_nu + g_nv_{2n-1} + h_nJ^{G_2}_{\sigma_n}(I - \sigma_nA_2^*(I - S)A_2)v_{2n-1}, \ n \in \mathbb{N}. \end{cases}$$

Then $\lim_{n\to\infty} v_n = P_{\Omega_{D1}}u$ provided the following conditions are satisfied:

(i) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} f_n = 0$;

(ii) *either* $\sum_{n=1}^{\infty} a_n = \infty$ *or* $\sum_{n=1}^{\infty} f_n = \infty$;

(iii) $0 < \rho_n < \frac{1}{\|A_1\|^2+2}$, $0 < \sigma_n < \frac{1}{\|A_2\|^2+2}$ *for each* $n \in \mathbb{N}$;

(iv) $\liminf_{n\to\infty} c_n > 0$ *and* $\liminf_{n\to\infty} h_n > 0$.

Proof. Put $F_1 = \frac{I+T}{2}$ and $F_2 = \frac{I+S}{2}$. Since T is a nonexpansive mappings of $\mathcal{H}_2$ into $\mathcal{H}_2$ and S is a nonexpansive mapping of $\mathcal{H}_3$ into $\mathcal{H}_3$. It is easy to see that F_1 and F_2 are firmly nonexpansive mappings. It is easy to see that algorithm in Theorem 4.1 follows immediately from algorithm in Theorem 3.7, $Fix(F_1) = Fix(T)$ and $Fix(F_2) = Fix(S)$. Therefore, Theorem 4.1 follows immediately from Theorem 3.7. □

For each $i = 1, 2, \ldots, m_1$ and $j = 1, 2, \ldots, m_2$, let T_i and S_j are averaged. We study the following multiple sets split feasible problem (**MSSFP-D2**):

Find $\bar{x} \in \mathcal{H}_1$ such that $\bar{x} \in G_1^{-1}(0) \cap G_2^{-1}(0)$, $A_1\bar{x} \in \bigcap_{i=1}^{m} Fix(T_i)$ and $A_2\bar{x} \in \bigcap_{j=1}^{n} Fix(S_j)$.

Theorem 4.2. *For each $i = 1, 2, \ldots, m_1$ and $j = 1, 2, \ldots, m_2$, let T_i and S_j are averaged. Suppose that the solutions set Ω_{D2} of $(\mathbf{MSSFP-D2})$ is nonempty, and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{f_n\}$, $\{g_n\}$, $\{h_n\}$ are sequences in $[0,1]$ with $a_n + b_n + c_n = 1$, $f_n + g_n + h_n = 1$, $0 < a_n < 1$, and $0 < f_n < 1$ for each $n \in \mathbb{N}$. For an arbitrary fixed $u \in \mathcal{H}$, a sequence $\{v_n\}$ be defined by*

$$\begin{cases} v_{2n+1} := a_nu + b_nv_{2n} + c_nJ^{G_1}_{\rho_n}(I - \rho_nA_1^*(I - T_1T_2T_3\ldots T_{m_1})A_1)v_{2n}, \\ \hfill n \in \mathbb{N} \cup \{0\}, \\ v_{2n} := f_nu + g_nv_{2n-1} + h_nJ^{G_2}_{\sigma_n}(I - \sigma_nA_2^*(I - S_1S_2\ldots S_{m_2})A_2)v_{2n-1}, \\ \hfill n \in \mathbb{N}. \end{cases}$$

Then $\lim_{n\to\infty} v_n = P_{\Omega_{D2}} u$ *provided the following conditions are satisfied:*

(i) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} f_n = 0$;
(ii) *either* $\sum_{n=1}^{\infty} a_n = \infty$ *or* $\sum_{n=1}^{\infty} f_n = \infty$;
(iii) $0 < \rho_n < \frac{1}{\|A_1\|^2+2}$, $0 < \sigma_n < \frac{1}{\|A_2\|^2+2}$ *for each* $n \in \mathbb{N}$;
(iv) $\liminf_{n\to\infty} c_n > 0$ *and* $\liminf_{n\to\infty} h_n > 0$.

Proof. Since for each $i = 1, 2, \ldots, m_1$ and $j = 1, 2, \ldots, m_2$, T_i and S_j are averaged. By Lemma 2.4, we knows that $T = T_1T_2\cdots T_{m_1}$ and $S = S_1S_2\cdots S_{m_2}$ are averaged. This shows that $T = T_1T_2\cdots T_{m_1}$ and $S = S_1S_2\cdots S_{m_2}$ are nonexpansive. By assumption, $\Omega_{D2} \neq \emptyset$, hence there exists $w \in \mathcal{H}_1$ such that $w \in G_1^{-1}(0) \cap G_2^{-1}(0)$, $A_1 w \in \bigcap_{i=1}^{m} Fix(T_i)$ and $A_2 w \in \bigcap_{j=1}^{n} Fix(S_j)$. By Lemma 2.4, $w \in G_1^{-1}(0) \cap G_2^{-1}(0)$, $A_1 w \in Fix(T)$ and $A_2 w \in Fix(S)$ and $\Omega_{D1} \neq \emptyset$. By Theorem 4.1, there exists $\bar{x} \in \mathcal{H}_1$ such that $\bar{x} \in G_1^{-1}(0) \cap G_2^{-1}(0)$, $A_1\bar{x} \in Fix(T_1T_2\cdots T_{m_1})$ and $A_2\bar{x} \in Fix(S_1S_2\cdots S_{m_2})$. By Lemma 2.4, $Fix(T_1T_2\cdots T_{m_1}) = \bigcap_{i=1}^{m} Fix(T_i)$ and $Fix(S_1S_2\cdots S_{m_2}) = \bigcap_{j=1}^{n} Fix(S_j)$. Therefore, the proof is completed. □

Let C, Q and Q' be nonempty closed convex subsets of $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$, respectively. For each $i = 1, 2, \ldots, 2m_1$, $j = 1, 2, \ldots, 2m_2$, $\kappa_i > 0$, and $\kappa'_j > 0$, let L_i be a κ_i-inverse-strongly monotone mapping of Q into H_2 and L'_j be a κ'_j-inverse-strongly monotone mapping of Q' into H_3. For each $i = 1, 2, \ldots, 2m_1$, $j = 1, 2, \ldots, 2m_2$, let M_i be a maximal monotone mapping on H_2 and M'_j be a be a maximal monotone mapping on H_3 such that the domain of M_i is included in Q, the domain of M'_j is included in Q'. We define the set $M_i^{-1}0$ as $M_i^{-1}0 = \{x \in H_i : 0 \in M_i x\}$. Let $J_{\lambda_n}^{M_i} = (I + \lambda_n M_i)^{-1}$ and $J_{r_n}^{M'_j} = (I + r_n M'_j)^{-1}$ for each $n \in \mathbb{N}$, $\lambda_n > 0$ and $r_n > 0$. Throughout this section and next section, we use these notations and assumptions unless specified otherwise.

In the following theorem, we study the following multiple sets split systems of variational inequalities problem (**MSSFP–D3**):

Find $\bar{x} \in \mathcal{H}_1$, such that for each $i = 1, 3, \ldots, 2m_1 - 1$, $j = 1, 3, \ldots, 2m_2 - 1$, thee exist $\bar{u}_i \in \mathcal{H}_2$, $\bar{w}_j \in \mathcal{H}_3$, with

(i) $\bar{x} \in G_1^{-1}(0) \cap G_2^{-1}(0)$;
(ii) $\langle \sigma L_{i+1} A_1\bar{x} + \bar{u}_i - A_1\bar{x}, u_i - \bar{u}_i \rangle \geq 0$ for all $u_i \in Fix J_\sigma^{M_{i+1}}$;
(iii) $\langle \sigma L_i \bar{u}_i + A_1\bar{x} - \bar{u}_i, y - A_1\bar{x} \rangle \geq 0$ for all $y \in Fix J_\sigma^{M_i}$;
(iv) $\langle \sigma L'_{j+1} A_2\bar{x} + \bar{w}_j - A_2\bar{x}, w_j - \bar{w}_j \rangle \geq 0$ for all $w_j \in Fix J_\sigma^{M'_{j+1}}$; and
(v) $\langle \sigma L'_j \bar{w}_j + A_2\bar{x} - \bar{w}_j, z - A_2\bar{x} \rangle \geq 0$ for all $z \in Fix J_\sigma^{M'_j}$.

Theorem 4.3. *Suppose that the solutions set* Ω_{D3} *of* (**MSSFP − D3**) *is nonempty, and* $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{f_n\}$, $\{g_n\}$, $\{h_n\}$ *are sequences in* $[0, 1]$ *with* $a_n + b_n + c_n = 1$, $f_n + g_n + h_n = 1$, $0 < a_n < 1$, *and* $0 < f_n < 1$ *for each* $n \in \mathbb{N}$. *For an arbitrary* *fixed* $u \in \mathcal{H}$*, a sequence* $\{v_n\}$ *be defined by*

$$\begin{cases} v_{2n+1} := & a_n u + b_n v_{2n} + c_n J_{\rho_n}^{G_1}(I - \rho_n A_1^*(I - J_\sigma^{M_1}(I - \sigma L_1) J_\sigma^{M_2}(I - \sigma L_2) \\ & \cdots J_\sigma^{M_{2m_1-1}}(I - \sigma L_{2m_1-1}) J_\sigma^{M_{2m_1}}(I - \sigma L_{2m_1}))A_1)v_{2n},\ n \in \mathbb{N} \cup \{0\}, \\ v_{2n} := & f_n u + g_n v_{2n-1} + h_n J_{\sigma_n}^{G_2}(I - \sigma_n A_2^*(I - J_\delta^{M_1^{'}}(I - \delta L_1^{'}) J_\delta^{M_2^{'}}(I - \delta L_2^{'}) \\ & \cdots J_\delta^{M_{2m_2-1}^{'}}(I - \delta L_{2m_2-1}^{'}) J_\delta^{M_{2m_2}^{'}}(I - \delta L_{2m_2}^{'}))A_2)v_{2n-1},\ n \in \mathbb{N}. \end{cases}$$

Then $\lim_{n\to\infty} v_n = P_{\Omega_{D3}} u$ *provided the following conditions are satisfied:*

(i) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} f_n = 0$*;*
(ii) *either* $\sum_{n=1}^{\infty} a_n = \infty$ *or* $\sum_{n=1}^{\infty} f_n = \infty$*;*
(iii) $0 < \rho_n < \frac{1}{\|A_1\|^2+2}$, $0 < \sigma_n < \frac{1}{\|A_2\|^2+2}$ *for each* $n \in \mathbb{N}$,
$0 < \sigma < 2\min\{\kappa_1, \kappa_2, \ldots, \kappa_{2m_1}\}$ *and* $0 < \delta < 2\min\{\kappa_1', \kappa_2', \ldots, \kappa_{2m_2}'\}$*;*
(iv) $\liminf_{n\to\infty} c_n > 0$ *and* $\liminf_{n\to\infty} h_n > 0$.

Proof. For each $i = 1, 3, .., 2m_1 - 1$ and $j = 1, 3, \ldots, 2m_2 - 1$, put $T_i = J_\sigma^{M_i}(I - \sigma L_i) J_\sigma^{M_{i+1}}(I - \sigma L_{i+1})$ and $S_j = J_\delta^{M_j^{'}}(I - \delta L_j^{'}) J_\delta^{M_{j+1}^{'}}(I - \delta L_{j+1}^{'})$ in Theorem 4.2. By Lemma 3.4, for each $i = 1, 2, \ldots, m_1$ and $j = 1, 2, \ldots, m_2$, $J_\sigma^{M_i}(I - \sigma L_i)$ and $J_\delta^{M_j^{'}}(I - \delta L_j^{'})$ are averaged. By Lemma 2.4, we see that T_i and S_j are averaged. Then, by Theorem 4.2, there exists $\bar{x} \in \mathcal{H}_1$ such that $\bar{x} \in G_1^{-1}(0) \cap G_2^{-1}(0)$, $A_1\bar{x} \in \bigcap_{i=1}^m Fix(T_i)$ and $A_2\bar{x} \in \bigcap_{j=1}^n Fix(S_i)$.

Put $A_1\bar{x} = \bar{y}$, and $A_2\bar{x} = \bar{z}$, then for each $i = 1, 3, \ldots, 2m_1 - 1$ and $j = 1, 3, \ldots, 2m_2 - 1$,

$$\bar{y} \in FixT_i = Fix(J_\sigma^{M_i}(I - \sigma L_i) J_\sigma^{M_{i+1}}(I - \sigma L_{i+1}))$$

and

$$\bar{z} \in FixS_j = Fix(J_\delta^{M_j^{'}}(I - \delta L_j^{'}) J_\delta^{M_{j+1}^{'}}(I - \delta L_{j+1}^{'})).$$

Therefore,

$$\bar{y} = J_\sigma^{M_i}(I - \sigma L_i) J_\sigma^{M_{i+1}}(I - \sigma L_{i+1})\bar{y}$$

and

$$\bar{z} = J_\delta^{M_j^{'}}(I - \delta L_j^{'}) J_\delta^{M_{j+1}^{'}}(I - \delta L_{j+1}^{'})\bar{z}.$$

For each $i = 1, 3, \ldots, 2m_1 - 1$ and $j = 1, 3, \ldots, 2m_2 - 1$, Put

$$\bar{u}_i = J_\sigma^{M_{i+1}}(I - \sigma L_{i+1})\bar{y}$$

and

$$\bar{w}_j = J_\delta^{M_{j+1}^{'}}(I - \delta L_{j+1}^{'})\bar{z}.$$

Then, for each $i = 1, 3, \ldots, 2m_1 - 1$ and $j = 1, 3, \ldots, 2m_2 - 1$,

$$\bar{y} = J_\sigma^{M_i}(I - \sigma L_i)\bar{u}_i$$

and

$$\bar{z} = J_\delta^{M_j'}(I - \delta L_j')\bar{w}_i.$$

By Lemma 2.2, for each $i = 1, 3, \ldots, 2m_1 - 1$ and $j = 1, 3, \ldots, 2m_2 - 1$, we obtain that

(i) $\langle \sigma L_{i+1}\bar{y} + \bar{u}_i - \bar{y}, u_i - \bar{u}_i\rangle \geq 0$ for all $u_i \in FixJ_\sigma^{M_{i+1}}$;

(ii) $\langle \sigma L_i\bar{u}_i + \bar{y} - \bar{u}_i, y - \bar{y}\rangle \geq 0$ for all $y \in FixJ_\sigma^{M_i}$;

(iii) $\langle \delta L_{j+1}'\bar{z} + \bar{w}_j - \bar{z}, w_j - \bar{w}_j\rangle \geq 0$ for all $w_j \in FixJ_\delta^{M_{j+1}'}$; and

(iv) $\langle \delta L_j'\bar{w}_j + \bar{z} - \bar{w}_j, z - \bar{z}\rangle \geq 0$ for all $z \in FixJ_\delta^{M_j'}$.

□

In the following theorem, we study the following split systems of variational inequalities problem (**MSSFP-D4**):

Find $\bar{x} \in \mathcal{H}_1$, such that for each $i = 1, 3, \ldots, 2m_1 - 1$, $j = 1, 3, \ldots, 2m_2 - 1$, thee exist $\bar{u}_i \in \mathcal{H}_2$, $\bar{w}_j \in \mathcal{H}_3$, with

(i) $\bar{x} \in G_1^{-1}(0) \cap G_2^{-1}(0)$,

(ii) $\langle \sigma L_{i+1}\bar{x} + \bar{u}_i - \bar{x}, u_i - \bar{u}_i\rangle \geq 0$ for all $u_i \in FixJ_\sigma^{M_{i+1}}$;

(iii) $\langle \sigma L_i\bar{u}_i + \bar{x} - \bar{u}_i, y - \bar{x}\rangle \geq 0$ for all $y \in FixJ_\sigma^{M_i}$;

(iv) $\langle \delta L_{j+1}'A_2\bar{x} + \bar{w}_j - A_2\bar{x}, w_j - \bar{w}_j\rangle \geq 0$ for all $w_j \in FixJ_\delta^{M_{j+1}'}$; and

(vi) $\langle \delta L_j'\bar{w}_j + A_2\bar{x} - \bar{w}_j, z - A_2\bar{x}\rangle \geq 0$ for all $z \in FixJ_\delta^{M_j'}$.

Theorem 4.4. *Suppose that the solutions set Ω_{D4} of* (**MSSFP − D4**) *is nonempty, and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{f_n\}$, $\{g_n\}$, $\{h_n\}$ are sequences in $[0,1]$ with $a_n+b_n+c_n = 1$, $f_n + g_n + h_n = 1$, $0 < a_n < 1$, and $0 < f_n < 1$ for each $n \in \mathbb{N}$. For an arbitrary fixed $u \in \mathcal{H}$, a sequence $\{v_n\}$ be defined by*

$$\begin{cases} v_{2n+1} := & a_n u + b_n v_{2n} + c_n J_{\rho_n}^{G_1}(I - \rho_n(I - J_\sigma^{M_1}(I - \sigma L_1)J_\sigma^{M_2}(I - \sigma L_2) \\ & \cdots J_\sigma^{M_{2m_1-1}}(I - \sigma L_{2m_1-1})J_\sigma^{M_{2m_1}}(I - \sigma L_{2m_1})))v_{2n},\ n \in \mathbb{N}\cup\{0\}, \\ v_{2n} := & f_n u + g_n v_{2n-1} + h_n J_{\sigma_n}^{G_2}(I - \sigma_n A_2^*(I - J_\delta^{M_1'}(I - \delta L_1')J_\delta^{M_2'}(I - \delta L_2') \\ & \cdots J_\delta^{M_{2m_2-1}'}(I - \delta L_{2m_2-1}')J_\delta^{M_{2m_2}'}(I - \delta L_{2m_2}'))A_2)v_{2n-1},\ n \in \mathbb{N}. \end{cases}$$

Then $\lim_{n\to\infty} v_n = P_{\Omega_{D4}}u$ provided the following conditions are satisfied:

(i) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} f_n = 0$;

(ii) *either* $\sum_{n=1}^\infty a_n = \infty$ *or* $\sum_{n=1}^\infty f_n = \infty$;

(iii) $0 < \rho_n < \frac{1}{3}$, $0 < \sigma_n < \frac{1}{\|A_2\|^2+2}$ *for each* $n \in \mathbb{N}$,
$0 < \sigma < 2\min\{\kappa_1, \kappa_2, \ldots, \kappa_{2m_1}\}$ *and* $0 < \delta < 2\min\{\kappa_1', \kappa_2', \ldots, \kappa_{2m_2}'\}$;

(iv) $\liminf_{n\to\infty} c_n > 0$ *and* $\liminf_{n\to\infty} h_n > 0$.

Proof. Put $A_1 = I$ in Theorem 4.3, then Theorem 4.4 follows immediately from Theorem 4.3. □

In the following theorem, we study the following multiple sets split systems of variational inequalities problem (**MSSFP–D5**):

Find $\bar{x} \in \mathcal{H}_1$, such that for each $i = 1, 3, \ldots, 2m_1 - 1$, $j = 1, 3, \ldots, 2m_2 - 1$, thee exist $\bar{u}_i \in \mathcal{H}_2$, $\bar{w}_j \in \mathcal{H}_3$, with

(i) $\bar{x} \in G_1^{-1}(0) \cap G_2^{-1}(0)$;
(ii) $\langle \sigma L_{i+1} A_1 \bar{x} + \bar{u}_i - A_1 \bar{x}, u_i - \bar{u}_i \rangle \geq 0$ for all $u_i \in Q$;
(iii) $\langle \sigma L_i \bar{u}_i + A_1 \bar{x} - \bar{u}_i, y - A_1 \bar{x} \rangle \geq 0$ for all $y \in Q$;
(iv) $\langle \delta L'_{j+1} A_2 \bar{x} + \bar{w}_j - A_2 \bar{x}, w_j - \bar{w}_j \rangle \geq 0$ for all $w_j \in Q'$; and
(v) $\langle \delta L'_j \bar{w}_j + A_2 \bar{x} - \bar{w}_j, z - A_2 \bar{x} \rangle \geq 0$ for all $z \in Q'$.

The following theorem is a special case of multiple set split systems of variational inequalities problem.

Theorem 4.5. *Suppose that the solutions set Ω_{D5} of* (**MSSFP − D5**) *is nonempty, and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{f_n\}$, $\{g_n\}$, $\{h_n\}$ are sequences in $[0,1]$ with $a_n + b_n + c_n = 1$, $f_n + g_n + h_n = 1$, $0 < a_n < 1$, and $0 < f_n < 1$ for each $n \in \mathbb{N}$. For an arbitrary fixed $u \in \mathcal{H}$, a sequence $\{v_n\}$ be defined by*

$$\begin{cases} v_{2n+1} := & a_n u + b_n v_{2n} \\ & + c_n J_{\rho_n}^{G_1}(I - \rho_n A_1^*(I - P_Q(I - \sigma L_1) P_Q(I - \sigma L_2) \\ & \cdots P_Q(I - \sigma L_{2m_1 - 1}) P_Q(I - \sigma L_{2m_1})) A_1) v_{2n}, \ n \in \mathbb{N} \cup \{0\}, \\ v_{2n} := & f_n u + g_n v_{2n-1} \\ & + h_n J_{\sigma_n}^{G_2}(I - \sigma_n A_2^*(I - P_{Q'}(I - \delta L'_1) P_{Q'}(I - \delta L'_2) \\ & \cdots P_{Q'}(I - \delta L'_{2m_2 - 1}) P_{Q'}(I - \delta L'_{2m_2})) A_2) v_{2n-1}, \ n \in \mathbb{N}. \end{cases}$$

Then $\lim_{n \to \infty} v_n = P_{\Omega_{D5}} u$ provided the following conditions are satisfied:

(i) *$\lim_{n \to \infty} a_n = \lim_{n \to \infty} f_n = 0$;*
(ii) *either $\sum_{n=1}^{\infty} a_n = \infty$ or $\sum_{n=1}^{\infty} f_n = \infty$;*
(iii) *$0 < \rho_n < \frac{1}{\|A_1\|^2 + 2}$, $0 < \sigma_n < \frac{1}{\|A_2\|^2 + 2}$ for each $n \in \mathbb{N}$, $0 < \sigma < \min\{\kappa_1, \kappa_2, \ldots, \kappa_{2m_1}\}$ and $0 < \delta < \min\{\kappa'_1, \kappa'_2, \ldots, \kappa'_{2m_2}\}$;*
(iv) *$\liminf_{n \to \infty} c_n > 0$ and $\liminf_{n \to \infty} h_n > 0$.*

Proof. For each $i = 1, 2, \ldots, 2m_1$ and $j = 1, 2, \ldots, 2m_2$, put $M_i = \partial i_Q$ and $M_j{}' = \partial i_{Q'}$. Then M_i is maximal monotone operator on $\mathcal{H}_2$, for each $i = 1, 2, \ldots, 2m_1$ and $M_J{}'$ is a maximal monotone operator on $\mathcal{H}_3$, for each $j = 1, 2, \ldots, 2m_2$. Since for each $i = 1, 2, \ldots, 2m_1$, $J_\sigma^{\partial i_Q} = P_Q$, $Fix(J_\sigma^{\partial i_Q}) = Q$ and for each $j = 1, 2, \ldots, 2m_2$, $J_{\sigma'}^{\partial i_{Q'}} = P_{Q'}$, $Fix(J_{\sigma'}^{\partial i_{Q'}}) = Q'$. Then Theorem 4.5 follows from Theorem 4.3. ☐

Remark 4.6. to the best of our knowledge, there is no result on the problems (**MSSFP − D2**, **MSSFP − D3**, **MSSFP − D4**, **MSSFP − D5**).

For each $i = 1, 2, \ldots, m_1$ and $j = 1, 2, \ldots, m_2$, let T_i and S_j be average. In the following theorem, we study the following convex feasibility problem (**MSSFP-D6**):

Find $\bar{x} \in \mathcal{H}_1$ such that $\bar{x} \in G_1^{-1}(0) \cap G_2^{-1}(0) \cap Fix(T_1) \cap Fix(T_2) \cap \cdots \cap Fix(T_{m_1}) \cap Fix(S_1) \cap Fix(S_2) \cap \cdots \cap Fix(S_{m_2})$.

The following theorem is a special case of Theorem 4.2, but it is useful to the study of other types of multiple sets split feasibility problems.

Theorem 4.7. *For each $i = 1, 2, \ldots, m_1$ and $j = 1, 2, \ldots, m_2$, let T_i and S_j be average. Suppose that the solutions set Ω_{D6} of* (**MSSFP − D6**) *is nonempty, and*

$\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{f_n\}$, $\{g_n\}$, $\{h_n\}$ *are sequences in* $[0,1]$ *with* $a_n + b_n + c_n = 1$, $f_n + g_n + h_n = 1$, $0 < a_n < 1$, *and* $0 < f_n < 1$ *for each* $n \in \mathbb{N}$. *For an arbitrary fixed* $u \in \mathcal{H}$, *a sequence* $\{v_n\}$ *be defined by*

$$\begin{cases} v_{2n+1} := & a_n u + b_n v_{2n} + c_n J_{\rho_n}^{G_1}(I - \rho_n(I - T_1T_2T_3\ldots T_{m_1}))v_{2n},\ n \in \mathbb{N}\cup\{0\}, \\ v_{2n} := & f_n u + g_n v_{2n-1} + h_n J_{\sigma_n}^{G_2}(I - \sigma_n(I - S_1S_2\ldots S_{m_2}))v_{2n-1},\ n \in \mathbb{N}. \end{cases}$$

Then $\lim_{n\to\infty} v_n = P_{\Omega_{D6}}u$ *provided the following conditions are satisfied:*

(i) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} f_n = 0$;
(ii) *either* $\sum_{n=1}^{\infty} a_n = \infty$ *or* $\sum_{n=1}^{\infty} f_n = \infty$;
(iii) $0 < \rho_n < \frac{1}{3}$, $0 < \sigma_n < \frac{1}{3}$ *for each* $n \in \mathbb{N}$,
(iv) $\liminf_{n\to\infty} c_n > 0$ *and* $\liminf_{n\to\infty} h_n > 0$.

Proof. Put $A_1 = I$ and $A_2 = I$ in Theorem 4.2, then Theorem 4.7 follows immediately from Theorem 4.2. □

5. Multiple sets split system of variational inclusion problems and multiple sets split systems of feasibility problems

For each $i \in \mathbb{N}$, $j \in \mathbb{N}$, let F_i be a firmly nonexpansive of $\mathcal{H}_2$ into $\mathcal{H}_2$, $F_j^{'}$ be a firmly nonexpansive mapping of $\mathcal{H}_3$ into $\mathcal{H}_3$. For each $i \in \mathbb{N}$, let $A_i : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator and A_i^* be the adjoint of A_i, for each $j \in \mathbb{N}$, let $A_j' : \mathcal{H}_1 \to \mathcal{H}_3$ be a bounded linear operator and $(A_j^{'})^*$ be the adjoint of A_j'. Throughout this section, we use these notations and assumptions unless specified otherwise.

In the following theorem, we study the multiple sets split system of variational inclusion problems (**MSSFP-E1**):

Find $\bar{x} \in \mathcal{H}_1$ such that $\bar{x} \in G_1^{-1}(0)\bigcap G_2^{-1}0$, $A_i\bar{x} \in (L_i + M_i)^{-1}0$ for each $i = 1, 2, \ldots, m_1$ and $A_j'\bar{x} \in (L_j' + M_j')^{-1}0$ for each $j = 1, 2, .., m_2$.

Theorem 5.1. *Suppose that the solutions set* Ω_{E1} *of* $(\mathbf{MSSFP} - \mathbf{E1})$ *is nonempty, and* $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{f_n\}$, $\{g_n\}$, $\{h_n\}$ *are sequences in* $[0,1]$ *with* $a_n + b_n + c_n = 1$, $f_n + g_n + h_n = 1$, $0 < a_n < 1$, *and* $0 < f_n < 1$ *for each* $n \in \mathbb{N}$.

For an arbitrary fixed $u \in \mathcal{H}$, *a sequence* $\{v_n\}$ *be defined by*

$$\begin{cases} v_{2n+1} := & a_n u + b_n v_{2n} \\ & +c_n J_{\rho_n}^{G_1}(I - \rho_n(I - (I - \sigma A_1^*(I - U_1)A_1)(I - \sigma A_2^*(I - U_2)A_2) \\ & \cdots (I - \sigma A_{m_1}^*(I - U_{m_1})A_{m_1}))v_{2n},\ m \in \mathbb{N}\cup\{0\}, \\ v_{2n} := & f_n u + g_n v_{2n-1} \\ & +h_n J_{\sigma_n}^{G_2}(I - \sigma_n(I - (I - \delta {A_1'}^*(I - U_1')A_1')(I - \delta {A_2'}^*(I - U_2')A_2') \\ & \cdots (I - \delta {A_{m_2}'}^*(I - U_{m_2}')A_{m_2}'))v_{2n-1},\ n \in \mathbb{N}. \end{cases}$$

where $U_i = J_\alpha^{M_i}(I - \alpha L_i)$ *and* $U_j^{'} = J_\beta^{M_j^{'}}(I - \beta L_j^{'})$.

Then $\lim_{n\to\infty} v_n = P_{\Omega_{E1}}u$ *provided the following conditions are satisfied:*

(i) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} f_n = 0$;
(ii) *either* $\sum_{n=1}^{\infty} a_n = \infty$ *or* $\sum_{n=1}^{\infty} f_n = \infty$;

(iii) $0 < \rho_n < \frac{1}{3}$, $0 < \sigma_n < \frac{1}{3}$ *for each* $n \in \mathbb{N}$, $0 < \alpha < 2\min\{\kappa_1, \ldots, \kappa_{m_1}\}$, $0 < \beta < 2\min\{\kappa_1^{'}, \ldots, \kappa_{m_2}^{'}\}$, $0 < \sigma < \min\{\frac{1}{R_1}, \frac{1}{R_2}, \ldots, \frac{1}{R_{m_1}}\}$ *and* $0 < \delta < \min\{\frac{1}{R_1'}, \frac{1}{R_2'}, \ldots, \frac{1}{R_{m_2}'}\}$;

(iv) $\liminf_{n\to\infty} c_n > 0$ $\liminf_{n\to\infty} h_n > 0$.

Proof. For each $i = 1, 2, \ldots, m_1$ and $j = 1, 2, \ldots, m_2$, let $T_i = (I - \sigma A_i^*(I - U_i)A_i)$ and $S_j = (I - \delta {A_j'}^*(I - U_j')A_j')$. By Lemma 3.4, for each $i = 1, 2, \ldots, m_1$ and $j = 1, 2, \ldots, m_2$, T_i and S_j are averaged. By Theorem 4.7, there exists $\bar{x} \in \mathcal{H}_1$ such that Find $\bar{x} \in \mathcal{H}_1$ such that $\bar{x} \in G_1^{-1}(0) \cap G_2^{-1}(0) \cap Fix(T_1) \cap Fix(T_2) \cap \cdots \cap Fix(T_{m_1}) \cap Fix(S_1) \cap Fix(S_2) \cap \cdots \cap Fix(S_{m_2})$. For each $i = 1, 2, \ldots, m_1$ and $j = 1, 2, \ldots, m_2$, $\bar{x} \in Fix(I - \sigma A_i^*(I - U_i)A_i)$ and $\bar{x} \in Fix(I - \delta {A_j'}^*(I - U_j')A_j')$. By Lemma 3.4, for each $i = 1, 2, \ldots, m_1$ and $j = 1, 2, \ldots, m_2$, $A_i\bar{x} \in FixU_i$, and $A_j{}'\bar{x} \in FixU_j^{'}$. Since $FixU_i = FixJ_\alpha^{M_i}(I - \alpha L_i) = (L_1 + M_i)^{-1}0$ and $FixU_j' = FixJ_\beta^{M_j^{'}}(I - \beta L_j^{'}) = (L_j^{'} + M_j^{'})^{-1}0$. □

Remark 5.2. There are some differences between Theorems 5.1 and 3.8. The multiple sets split monotone variational inclusion problem studies in Theorem 5.1 has system type, but Theorem 3.8 does not study system type.

In the following theorem, we study the multiple sets split system of variational inclusion problems (**MSSFP-E2**):

Find $\bar{x} \in \mathcal{H}_1$ such that $\bar{x} \in G_1^{-1}0 \bigcap G_2^{-1}0$, $A_i\bar{x} \in Fix(F_i)$ for each $i = 1, 2, \ldots, m$ and $A_j'\bar{x} \in Fix(F_j')$ for each $j = 1, 2, .., n$.

Theorem 5.3. *Suppose that the solutions set* Ω_{E2} *of* $(\mathbf{MSSFP - E2})$ *is nonempty, and* $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{f_n\}$, $\{g_n\}$, $\{h_n\}$, *are sequences in* $[0, 1]$ *with* $a_n + b_n + c_n = 1$, $f_n + g_n + h_n = 1$, $0 < a_n < 1$, *and* $0 < f_n < 1$ *for each* $n \in \mathbb{N}$. *For each* $i = 1.2, \ldots, m_1$ *and* $j = 1, 2, \ldots, m_2$. *For an arbitrary fixed* $u \in \mathcal{H}_1$, *a sequence* $\{v_n\}$ *be defined by*

$$\begin{cases} v_{2n+1} := & a_n u + b_n v_{2n} \\ & +c_n J_{\rho_n}^{G_1}(I - \rho_n(I - (I - \sigma A_1^*(I - F_1)A_1)(I - \sigma A_2^*(I - F_2)A_2) \\ & \cdots (I - \sigma A_{m_1}^*(I - F_{m_1})A_{m_1}))v_{2n},\ n \in \mathbb{N} \cup \{0\}, \\ v_{2n} := & f_n u + g_n v_{2n-1} \\ & +h_n J_{\sigma_n}^{G_2}(I - \sigma_n(I - (I - \delta {A_1^{'}}^*(I - F_1^{'})A_1^{'})(I - \delta {A_2^{'}}^*(I - F_2^{'})A_2^{'}) \\ & \cdots (I - \delta {A_{m_2}^{'}}^*(I - F_{m_2}^{'})A_{m_2}^{'}))v_{2n-1},\ n \in \mathbb{N}. \end{cases}$$

Then $\lim_{n\to\infty} v_n = P_{\Omega_{E2}}u$ *provided the following conditions are satisfied:*

(i) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} f_n = 0$;

(ii) *either* $\sum_{n=1}^\infty a_n = \infty$ *or* $\sum_{n=1}^\infty f_n = \infty$;

(iii) $0 < \rho_n < \frac{1}{3}$, $0 < \sigma_n < \frac{1}{3}$ *for each* $n \in \mathbb{N}$, $0 < \sigma < \min\left\{\frac{2}{\|A_1\|^2+2}, \frac{2}{\|A_2\|^2+2}, \ldots, \frac{2}{\|A_{m_1}\|^2+2}\right\}$ *and* $0 < \delta < \min\left\{\frac{2}{\|A_1'\|^2+2}, \frac{2}{\|A_2'\|^2+2}, \ldots, \frac{2}{\|A_{m_2}'\|^2+2}\right\}$;

(iv) $\liminf_{n\to\infty} c_n > 0$ *and* $\liminf_{n\to\infty} h_n > 0$.

Proof. For each $i = 1, 2, \dots, m_1$ and $j = 1, 2, \dots, m_2$, let $T_i = (I - \sigma A_i^*(I - F_i)A_i)$ and $S_j = (I - \delta {A'_j}^*(I - F'_j)A'_j)$. By Lemma 3.5, for each $i = 1, 2, \dots, m_1$ and $j = 1, 2, \dots, m_2$, T_i and S_j are averaged. By Theorem 4.7, there exists $\bar{x} \in \mathcal{H}_1$ such that $\bar{x} \in G_1^{-1}(0) \cap G_2^{-1}(0) \cap Fix(T_1) \cap Fix(T_2) \cap \cdots \cap Fix(T_m) \cap Fix(S_1) \cap Fix(S_2) \cap \cdots \cap Fix(S_n)$. For each $i = 1, 2, \dots, m_1$ and $j = 1, 2, \dots, m_2$, $\bar{x} \in Fix((I - \sigma A_i^*(I - F_i)A_i))$ and $\bar{x} \in Fix((I - \delta {A'_1}^*(I - F'_1)A'_1))$. By Lemma 3.5,, for each $i = 1, 2, \dots, m_1$ and $j = 1, 2, \dots, m_2$, $A_i\bar{x} \in Fix(F_i)$ and $A'_j\bar{x} \in Fix(F'_j)$. $\square$

In the following theorem, we study the multiple sets split system of variational inclusion problems (**MSSFP-E3**):

Find $\bar{x} \in \mathcal{H}_1$ such that $\bar{x} \in G_1^{-1}(0) \bigcap G_2^{-1}0$, $A_i\bar{x} \in (L_i + M_i)^{-1}0$ for each $i = 1, 2, \dots, m_1$ and $A'_j\bar{x} \in Fix(F'_j)$ for each $j = 1, 2, \dots, m_2$.

Remark 5.4. There are some differences between Theorems 5.3 and 3.7. The multiple sets split feasibility problem study in Theorem 5.3 has system type, but Theorem 3.7 does not study system type.

Theorem 5.5. *Suppose that the solutions set Ω_{E3} of* $(\mathbf{MSSFP-E3})$ *is nonempty, and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{f_n\}$, $\{g_n\}$, $\{h_n\}$, are sequences in $[0,1]$ with $a_n+b_n+c_n = 1$, $f_n + g_n + h_n = 1$, $0 < a_n < 1$, and $0 < f_n < 1$ for each $n \in \mathbb{N}$.*

For an arbitrary fixed $u \in \mathcal{H}_1$, a sequence $\{v_n\}$ be defined by

$$\begin{cases} v_{2n+1} := & a_n u + b_n v_{2n} \\ & +c_n J_{\rho_n}^{G_1}(I - \rho_n(I - (I - \sigma A_1^*(I - U_1)A_1)(I - \sigma A_2^*(I - U_2)A_2) \\ & \cdots (I - \sigma A_{m_1}^*(I - U_{m_1})A_{m_1}))v_{2n}, \ n \in \mathbb{N} \cup \{0\}, \\ v_{2n} := & f_n u + g_n v_{2n-1} \\ & +h_n J_{\sigma_n}^{G_2}(I - \sigma_n(I - (I - \delta {A_1'}^*(I - F_1')A_1')(I - \delta {A_2'}^*(I - F_2')A_2') \\ & \cdots (I - \delta {A_{m_2}'}^*(I - F_{m_2}')A_{m_2}'))v_{2n-1}, \ n \in \mathbb{N}. \end{cases}$$

where $U_i = J_\alpha^{M_i}(I - \alpha L_i)$. Then $\lim_{n\to\infty} v_n = P_{\Omega_{E3}}u$ provided the following conditions are satisfied:

(i) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} f_n = 0$;
(ii) *either* $\sum_{n=1}^\infty a_n = \infty$ *or* $\sum_{n=1}^\infty f_n = \infty$;
(iii) $0 < \rho_n < \frac{1}{3}$, $0 < \sigma_n < \frac{1}{3}$ *for each* $n \in \mathbb{N}$, $0 < \alpha < 2\min\{\kappa_1, \dots, \kappa_m\}$,
$\sigma < \min\left\{\frac{2}{\|A_1\|^2+2}, \frac{2}{\|A_2\|^2+2}, \dots, \frac{2}{\|A'_{m_1}\|^2+2}\right\}$ *and*
$0 < \delta < \min\left\{\frac{1}{R'_1}, \frac{1}{R'_2}, \dots, \frac{1}{R'_{m_2}}\right\}$;
(iv) $\liminf_{n\to\infty} c_n > 0$ *and* $\liminf_{n\to\infty} h_n > 0$.

Proof. For each $i = 1, 2, \dots, m_1$ and $j = 1, 2, \dots, m_2$, let $T_i = (I - \sigma A_i^*(I - U_i)A_i)$ and $S_j = (I - \delta {A'_j}^*(I - F'_j)A'_j)$. By Lemmas 3.4 and 3.5, for each $i = 1, 2, \dots, m_1$ and $j = 1, 2, \dots, m_2$, T_i and S_j are averaged. By Theorem 4.7, there exists $\bar{x} \in \mathcal{H}_1$ such that Find $\bar{x} \in \mathcal{H}_1$ such that $\bar{x} \in G_1^{-1}(0) \cap G_2^{-1}(0) \cap Fix(T_1) \cap Fix(T_2) \cap \cdots \cap Fix(T_{m_1}) \cap Fix(S_1) \cap Fix(S_2) \cap \cdots \cap Fix(S_{m_2})$. For each $i = 1, 2, \dots, m_1$ and $j = 1, 2, \dots, m_2$, $\bar{x} \in Fix(I - \sigma A_i^*(I - U_i)A_i))$ and $\bar{x} \in Fix(I - \delta {A'_1}^*(I - F'_1)A'_1)$. By Lemmas 3.4 and 3.5, for each $i = 1, 2, \dots, m_1$ and $j = 1, 2, \dots, m_2$, $\bar{x} \in G_1^{-1}0$, $A_i\bar{x} \in FixU_i$, $\bar{x} \in G_2^{-1}0$ and ${A_j}'\bar{x} \in FixF_j$ and we also know that $FixU_i =$

$Fix{J_\alpha}^{M_i}(I-\alpha L_i) = (L_1+M_i)^{-1}0$. Therefore, this complete the proof of Theorem 5.5. □

Remark 5.6. There are some differences between Theorems 5.5 and 3.10.

For each $i = 1,2,\dots,m_1$ and $j = 1,2,\dots,m_2$, let T_i and S_j are nonexpansive mappings. In the following theorem, we study the multiple sets split system of variational inclusion problems (**MSSFP-E4**):

Find $\bar{x} \in \mathcal{H}_1$ such that $\bar{x} \in G_1^{-1}0 \bigcap G_2^{-1}0$, $A_i\bar{x} \in Fix(T_i)$ for each $i = 1,2,\dots,m$ and $A'_j\bar{x} \in Fix(S_j)$ for each $j = 1,2,..,n$.

Theorem 5.7. *For each $i = 1,2,\dots,m_1$ and $j = 1,2,\dots,m_2$, let T_i and S_j are nonexpansive mappings. Suppose that the solutions set Ω_{E4} of* (**MSSFP − E4**) *is nonempty and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{f_n\}$, $\{g_n\}$, $\{h_n\}$ are sequences in $[0,1]$ with $a_n+b_n+c_n = 1$, $f_n+g_n+h_n = 1$, $0 < a_n < 1$, and $0 < f_n < 1$ for each $n \in \mathbb{N}$.*

For an arbitrary fixed $u \in \mathcal{H}_1$, a sequence $\{v_n\}$ be defined by

$$\begin{cases} v_{2n+1} := & a_nu + b_nv_{2n} \\ & +c_nJ_{\rho_n}^{G_1}(I-\rho_n(I-(I-\frac{1}{2}\sigma A_1^*(I-T_1)A_1)(I-\frac{1}{2}\sigma A_2^*(I-T_2)A_2) \\ & \cdots(I-\frac{1}{2}\sigma A_{m_1}^*(I-T_{m_1})A_{m_1})))v_{2n},\ n \in \mathbb{N}\cup\{0\}, \\ v_{2n} := & f_nu + g_nv_{2n-1} \\ & +h_nJ_{\sigma_n}^{G_2}(I-\sigma_n(I-J_\delta^{G_2}(I-\frac{1}{2}\delta {A'_1}^*(I-S_1)A'_1) \\ & J_\delta^{G_2}(I-\frac{1}{2}\delta {A'_2}^*(I-S_2)A'_2)\cdots J_\delta^{G_2}(I-\frac{1}{2}\delta {A'_{m_2}}^*(I-S_{m_2})A'_{m_2}))v_{2n-1}, \\ & \hfill n \in \mathbb{N} \end{cases}$$

Then $\lim_{n\to\infty} v_n = P_{\Omega_{E4}}u$ provided the following conditions are satisfied:

(i) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} f_n = 0$;
(ii) *either* $\sum_{n=1}^\infty a_n = \infty$ *or* $\sum_{n=1}^\infty f_n = \infty$;
(iii) $0 < \rho_n < \frac{1}{3}$, $0 < \sigma_n < \frac{1}{3}$ *for each* $n \in \mathbb{N}$,
$0 < \sigma < \min\left\{\frac{2}{\|A_1\|^2+2}, \frac{2}{\|A_2\|^2+2}, \dots, \frac{2}{\|A_{m_1}\|^2+2}\right\}$ *and*
$0 < \delta < \min\left\{\frac{2}{\|A'_1\|^2+2}, \frac{2}{\|A'_2\|^2+2}, \dots, \frac{2}{\|A'_{m_2}\|^2+2}\right\}$;
(iv) $\liminf_{n\to\infty} c_n > 0$ *and* $\liminf_{n\to\infty} h_n > 0$.

Proof. For each $i = 1,2,\dots,m_1$ and $j = 1,2,\dots,m_2$, let $F_i = \frac{I+T_1}{2}$ and $F'_j = \frac{I+S_j}{2}$ in Theorem 5.3. Applying Theorem 5.3 and following the same argument as in Theorem 4.1, we can prove Theorem 5.7. □

Remark 5.8. There are some differences between Theorems 5.5 and 3.10. To the best of our knowledge, there is no results on the problem (**MSSFP-E1, MSSFP-E2, MSSFP-E3, MSSFP-E4 and MSSFP-E5**).

In the following theorem, we study the following multiple sets split systems of variational inequalities problem (**MSSFP–E5**):

Find $\bar{x} \in \mathcal{H}_1$, such that for each $i = 1,2,\dots,2m_1$, $j = 1,2,\dots,2m_2$, thee exist $\bar{u}_i \in \mathcal{H}_2$, $\bar{w}_{j+1} \in \mathcal{H}_3$, with

(i) $\bar{x} \in G_1^{-1}(0) \cap G_2^{-1}(0)$;
(ii) $\langle \sigma L_{2i}A_i\bar{x} + \bar{u}_i - A_i\bar{x}, u_i - \bar{u}_i\rangle \geq 0$ for all $u_i \in FixJ_\sigma^{M_{2i}}$;

(iii) $\langle \sigma L_{2i-1}\bar{u}_i + A_i\bar{x} - \bar{u}_i, u_{2i-1} - A_i\bar{x}\rangle \geq 0$ for all $u_{2i-1} \in FixJ_\sigma^{M_{2i-1}}$;

(iv) $\langle \delta L'_{2j}A'_j\bar{x} + \bar{w}_j - A'_j\bar{x}, w_j - \bar{w}_j\rangle \geq 0$ for all $w_j \in FixJ_\delta^{M'_{2j}}$; and

(v) $\langle \delta L'_{2j-1}\bar{w}_j + A'_j\bar{x} - \bar{w}_j, w_{J+1} - A'_j\bar{x}\rangle \geq 0$ for all $w_{j+1} \in FixJ_\delta^{M'_{2j-1}}$.

Theorem 5.9. *Suppose that the solutions set Ω_{E5} of* (**MSSFP − E5**) *is nonempty, and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{f_n\}$, $\{g_n\}$, $\{h_n\}$ are sequences in $[0,1]$ with $a_n+b_n+c_n = 1$, $f_n + g_n + h_n = 1$, $0 < a_n < 1$, and $0 < f_n < 1$ for each $n \in \mathbb{N}$. For an arbitrary fixed $u \in \mathcal{H}_1$, a sequence $\{v_n\}$ be defined by*

$$\begin{cases} v_{2n+1} := & a_n u + b_n v_{2n} \\ & +c_n J_{\rho_n}^{G_1}(I - \rho_n(I - (I - \frac{1}{2}\sigma A_1^*(I - J_\sigma^{M_1}(I - \sigma L_1)J_\sigma^{M_2}(I - \sigma L_2))A_1) \\ & (I - \frac{1}{3}\sigma A_2^*(I - J_\sigma^{M_3}(I - \sigma L_3)J_\sigma^{M_4}(I - \sigma L_4))A_2)\cdots \\ & (I - \frac{1}{2}\sigma A_{m_1}^*(I - J_\sigma^{M_{2m_1-1}}(I - \sigma L_{2m_1-1})J_\sigma^{M_{2m_1}}(I - \sigma L_{2m_1}))A_{m_1}))v_{2n}, \\ & \qquad n \in \mathbb{N}\cup\{0\}, \\ v_{2n} := & f_n u + g_n v_{2n-1} \\ & +h_n J_{\sigma_n}^{G_2}(I - \sigma_n(I - (I - \frac{1}{2}\delta {A'_1}^*(I - J_\delta^{M_1'}(I - \delta L_1')J_\delta^{M_2'}(I - \delta L_2'))A'_1) \\ & (I - \frac{1}{2}\delta {A'_2}^*(I - J_\delta^{M_3'}(I - \delta L_3')J_\delta^{M_4'}(I - \delta L_4'))A'_2)\cdots(I - \frac{1}{2}\delta {A'_{m_2}}^* \\ & (I - J_\delta^{{M_{2m_2-1}}'}(I - \delta {L_{2m_2-1}}')J_\delta^{{M_{2m_2}}'}(I - \delta {L_{2m_2}}'))A'_{m_2}))v_{2n-1}, n \in \mathbb{N} \end{cases}$$

Then $\lim_{n\to\infty} v_n = P_{\Omega_{E5}}u$ provided the following conditions are satisfied:

(i) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} f_n = 0$;

(ii) *either* $\sum_{n=1}^\infty a_n = \infty$ *or* $\sum_{n=1}^\infty f_n = \infty$;

(iii) $0 < \rho_n < \frac{1}{3}$, $0 < \sigma_n < \frac{1}{3}$ *for each* $n \in \mathbb{N}$, $0 < \sigma < \min\left\{\frac{2}{\|A_1\|^2+2}, \frac{2}{\|A_2\|^2+2}, \ldots, \frac{2}{\|A_{m_1}\|^2+2}\right\}$ *and* $0 < \delta < \min\left\{\frac{2}{\|A'_1\|^2+2}, \frac{2}{\|A'_2\|^2+2}, \ldots, \frac{2}{\|A'_{m_2}\|^2+2}\right\}$; $0 < \sigma < 2\min\{\kappa_1, \kappa_2, \ldots, \kappa_{2m_1}\}$ *and* $0 < \delta < 2\min\{\kappa'_1, \kappa'_2, \ldots, \kappa'_{2m_2}\}$;

(iv) $\liminf_{n\to\infty} c_n > 0$ *and* $\liminf_{n\to\infty} h_n > 0$.

Proof. Applying Theorem 5.7 and following the same arguments as in the proof of Theorem 4.3, we can prove Theorem 5.9. □

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